Differentiating Matrix Functions

Abstract:

If $M_n(\mathbb{C})$ is the set of $n \times n$ complex matrices and $A \in M_n(\mathbb{C})$, then we write $\sigma(A) \subseteq \mathbb{C}$ for the set of eigenvalues of $A$. If $A$ is diagonalizable and $f: \sigma(A) \to \mathbb{C}$ is any function, then one can define $f(A) \in M_n(\mathbb{C})$ in a reasonable way. Now, let $M_n(\mathbb{C})_{sa}$ be the set of $n \times n$ Hermitian matrices, which are unitarily diagonalizable and have real eigenvalues. If $f: \mathbb{R} \to \mathbb{C}$ is a continuous function, then one can fairly easily show that the map $\tilde{f}: M_n(\mathbb{C})_{sa} \to M_n(\mathbb{C})$ defined by $A \mapsto f(A)$ is also continuous. In this talk, we shall discuss the less elementary fact that if $f$ is $k$-times continuously differentiable, then so is $\tilde{f}$. Time permitting, we shall also discuss the much more complicated infinite-dimensional case – where instead of matrices, one considers linear operators on a Hilbert space – which is still an active area of research.