1. (12 points) Find the following indefinite integrals.

(a) $\int \cos(\theta)\sqrt{\sin(\theta)+2}\,d\theta$

$u = \sin\theta + 2$

$du = \cos\theta \, d\theta$

$= \int [\sin\theta + 2]^{1/2} \cdot \underbrace{\cos\theta \, d\theta}_{du}$

$= \int u^{1/2} \, du$

$= \frac{2}{3} u^{3/2} + C$

$= \frac{2}{3}(\sin\theta + 2)^{3/2} + C$

(b) $\int \frac{u}{u^2+u}\,du$

$= \int \frac{u}{u(u+1)} \, du = \int \frac{1}{u+1} \, du = \ln|u+1| + C$

(c) $\int x^2 \ln(2x)\,dx$ $= \int \underbrace{\ln(2x)}_{u} \cdot \underbrace{x^2 dx}_{dv}$

$u = \ln(2x)$ $\quad dv = x^2 dx$

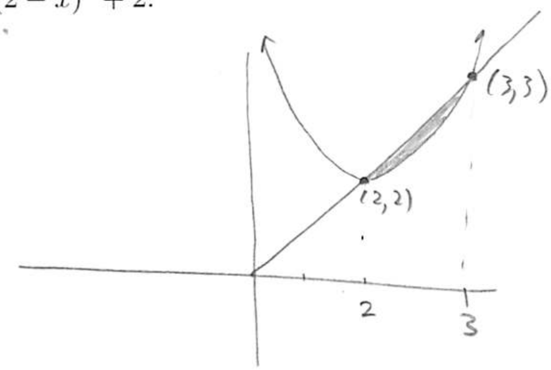
$du = \frac{1}{x} dx$ $\quad v = x^3/3$

$= \frac{x^3}{3} \ln(2x) - \int \frac{x^3}{3} \cdot \frac{1}{x} \, dx$

$= \frac{x^3}{3} \ln(2x) - \int \frac{x^2}{3} dx$

$= \frac{x^3}{3} \ln(2x) - \frac{x^3}{9} + C$

2. (6 points) Find the finite area enclosed by $y = x$ and $y = (2-x)^2 + 2$.



Find points of intersection:

$$x = (2-x)^2 + 2$$

$$x = 4 - 4x + x^2 + 2$$

$$x^2 - 5x + 6 = 0$$

$$(x-2)(x-3) = 0 \qquad \boxed{x=2,\ x=3}$$

$$\text{Area} = \int_2^3 \left(x - \left[(2-x)^2 + 2 \right] \right) dx$$

$$= \int_2^3 \left(x - (4 - 4x + x^2 + 2) \right) dx = \int_2^3 \left(-x^2 + 5x - 6 \right) dx$$

$$= -\frac{x^3}{3} + \frac{5x^2}{2} - 6x \Big|_2^3$$

$$= \left(-3^2 + \frac{5}{2} \cdot 3^2 - 2 \cdot 3^2 \right) - \left(-\frac{8}{3} + \underbrace{10 - 12}_{-2 = \frac{-6}{3}} \right)$$

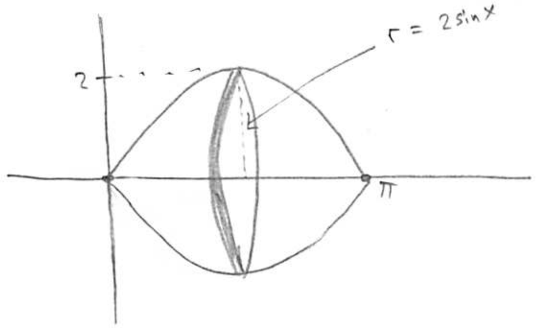
$$= 3^2 \left(\underbrace{-1 + \frac{5}{2} - 2}_{-3 + \frac{5}{2} = -\frac{6}{2} + \frac{5}{2}} \right) - \left(-\frac{14}{3} \right)$$

$$= 9\left(-\frac{1}{2} \right) + \frac{14}{3}$$

$$= -\frac{9}{2} + \frac{14}{3}$$

$$= \frac{-27 + 28}{6} = \frac{1}{6}$$

3. (6 points) Find the volume of the solid that results from revolving about the x-axis the region enclosed by the x-axis and the curve $y = 2\sin(x)$ for $0 \le x \le \pi$.



Volume of slice $\approx \pi r^2$

$= \pi (2\sin x)^2 = 4\pi \sin^2 x$

$$\text{Volume} = \int_0^{\pi} 4\pi \sin^2 x \, dx = \int_0^{\pi} 4\pi \cdot \tfrac{1}{2}(1 - \cos(2x))\, dx$$

$$= 2\pi \left(x - \tfrac{1}{2}\sin(2x) \right) \Big|_0^{\pi}$$

$$= 2\pi \left((\pi - 0) - (0 - 0) \right) = \boxed{2\pi^2}$$

4. (6 points) Determine the solution to the following initial value problem

$$\begin{cases} \dfrac{dy}{dx} = y + yx^2 \\ \\ y(0) = 3 \end{cases}$$

$$\frac{dy}{dx} = y(1+x^2)$$

$$\int \frac{dy}{y} = \int (1+x^2)\,dx$$

$$\ln|y| = x + \frac{x^3}{3} + c$$

$$|y| = e^{x + x^3/3 + c} = e^{x+x^3/3} e^c$$

$$y = \underbrace{(\pm e^c)}_{=A} e^{x+x^3/3} = A e^{x+x^3/3}$$

$$\begin{cases} y = A e^{x+x^3/3} \\ y(0) = 3 \end{cases} \Longrightarrow \quad y(0) = A e^0 = A(1) = A = 3$$

$$\boxed{y(t) = 3e^{x+x^3/3}}$$

5. (8 points) Determine whether each improper integral converges or diverges; if it converges, find its value.

(a) $\displaystyle\int_{-2}^{6} \frac{1}{(t+2)^{2/3}}\,dt$

$u = t+2 \qquad du = dt$

$t = -2 \qquad t = 6$

$u = 0 \qquad u = 8$

$$= \int_0^8 \frac{1}{u^{2/3}}\,du$$

$$= \lim_{a\to 0^+} \int_a^8 u^{-2/3}\,du$$

$$= \lim_{a\to 0^+} 3u^{1/3}\Big|_a^8 = \lim_{a\to 0^+} \underbrace{3(8)^{1/3}}_{=2} - 3a^{1/3}$$

$$= 3(2) - 0 = \boxed{6}$$

(converges)

(b) $\displaystyle\int_{15}^{\infty} \frac{2}{x\ln(x)}\,dx$

$u = \ln x \qquad du = \frac{1}{x}\,dx$

$x = 15 \qquad x \to \infty$

$u = \ln 15 \qquad u \to \infty$

$$= \int_{15}^{\infty} \frac{2}{\ln x} \cdot \frac{1}{x}\,dx$$

$$= 2\int_{\ln 15}^{\infty} \frac{1}{u}\,du$$

Known to diverge $(p=1)$

6. A company earns revenue at a continuous annual rate of 0.04 (4%) of its net worth. At the same time, the company's payroll obligations amount to 10 million dollars per year, paid out continuously.

(a) (2 points) Write a differential equation that governs the net worth, W, of the company in millions of dollars.

$$\frac{dW}{dt} = 0.04W - 10$$

(b) (2 points) Determine the initial net worth W_0 that will keep the net worth constant; in other words, determine the value of W_0 for which $W = W_0$ is an equilibrium solution of the differential equation.

$$\frac{dW}{dt} = 0 \implies W_0 = \frac{10}{0.04} = 250$$

(c) (4 points) Solve the differential equation given that the initial net worth is $W_0 = 100$.

$$\boxed{\text{Note: } 0.04 = \tfrac{1}{25}}$$

$$\int \frac{dw}{0.04W-10} = \int dt$$

$$\frac{1}{0.04} \ln|0.04W-10| = t + C \implies \ln|0.04w-10| = \frac{1}{25}(t+c)$$

$$0.04W - 10 = \pm e^{t/25} e^{C/25} = (\pm e^{c/25}) e^{t/25} = Ae^{t/25}$$

$$W = \frac{Ae^{t/25} + 10}{0.04} = Be^{t/25} + 250$$

$$W_0 = W(0) = B + 250 = 100 \implies B = -150$$

$$\boxed{W = -150 e^{t/25} + 250}$$

(d) (2 points) How long will it take for the company to go bankrupt? In other words, how long will it take for the net worth of the company to be zero?

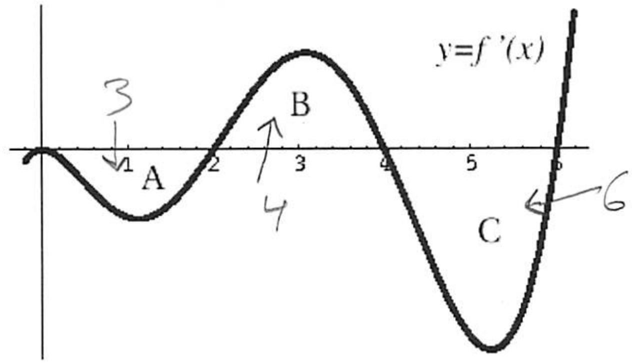
$$W = 0$$

$$-150 e^{t/25} + 250 = 0$$

$$e^{t/25} = \frac{250}{150} = \frac{25}{15} = \frac{5}{3}$$

$$\boxed{t = 25 \ln(5/3) \text{ years}}$$

7. Let $f(x)$ be a function satisfying $f(0) = 2$ whose ***derivative*** is graphed below; the area of region **A** is **3**, the area of region **B** is **4**, and the area of region **C** is **6**.



(a) (2 points) Find the value of $f(6)$.

$$f(6) - f(0) = \int_0^6 f'(x)\,dx$$

$$f(6) = \int_0^6 f'(x)\,dx + f(0)$$

$$= \underbrace{(-3+4-6)}_{=-5} + (2) = \boxed{-3}$$

(b) (2 points) Find the average rate of change of f on the interval $[0, 6]$.

$$= \frac{1}{6}\int_0^6 f'(x)\,dx = \frac{1}{6}(-5) = \boxed{-\frac{5}{6}}$$

(c) (4 points) Find the absolute (global) maximum value of $f(x)$ on the interval $[0, 6]$, and state where it occurs.

check endpoints and critical points

endpoints: $x=0$, $x=6$

critical points: $x=0$, $x=2$, $x=4$, $x=6$

$f(0) = 2$

$f(2) = 2-3 = -1$

$\boxed{f(4) = -1+4 = 3}$

$f(6) = 3-6 = -3$

Global maximum value is 3 at $x = 4$