Example: A pair of fair six-sided dice are rolled until the numbers they show sum to 7. Let X denote the number of rolls that are needed to get a sum of 7. Let Y be a random variable that denotes the number of rolls that had a sum of 2, 3, or 12. Compute the *joint PMF* for X an Y.

Solution: We wish to compute $p_{X,Y}(j,k) = P(X = j, Y = k)$. Observe first that $p_{X,Y}(j,k) = 0$ if $j < k$. Suppose then that $j > k$. By the Multiplication Rule:

$$p_{X,Y}(j,k) = P(X = j, Y = k) = \underbrace{P(X = j)}_{(1)} \cdot \underbrace{P(Y = k | X = j)}_{(2)}$$

We will compute each of these probabilities separately.

(1) The random variable X counts how many trials (rolls) are required the first success (sum of 7). Therefore, $X \sim \text{Geom}(p)$, where $p = P(\text{sum of } 7) = \frac{6}{36} = \frac{1}{6}$. Thus,

$$P(X = j) = \left(\frac{5}{6}\right)^{j-1}\left(\frac{1}{6}\right).$$

(2) The random variable Y counts the number of trials (rolls) that result is a success (sum of 2, 3, or 12). We are assuming that $X = j$, that means there are a total of j rolls, none of which result in a sum of 7 except for the last one. (So, in particular, the last roll cannot be 2, 3, or 12.) Therefore, $Y|(X = j)$ is a binomial random variable with number of trials $n = j - 1$ and probability of success

$$p = P(\text{sum of 2, 3, or 12} \mid \text{sum is not 7}) = \frac{4}{30} = \frac{2}{15}.$$

(Since the sum was not 7, there were only $36 - 6 = 30$ possible outcomes.) Consequently,

$$P(Y = k | X = j) = \binom{j-1}{k}\left(\frac{2}{15}\right)^{k}\left(\frac{13}{15}\right)^{j-1-k}.$$

Therefore,

$$p_{X,Y}(j,k) = \left(\frac{5}{6}\right)^{j-1}\left(\frac{1}{6}\right)\binom{j-1}{k}\left(\frac{2}{15}\right)^{k}\left(\frac{13}{15}\right)^{j-1-k},$$

if $j > k$, and zero otherwise.