A FUNDAMENTAL THEOREM OF INVARIANT METRICS ON A HOMOGENEOUS SPACE

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1. Introduction

A homogeneous space is a differentiable manifold M upon which a Lie group acts transitively. At the foundation of the study of homogeneous spaces is the following well-known theorem (e.g., Theorem 9.3 in [1], or Theorem 3.62 in [4]):

Theorem 1.1. *Let G be a Lie group acting transitively on a manifold M by a smooth action μ (i.e., M is a homogeneous space). If x is any point in M, then the mapping $F : G/G_x \to M$ defined by $F(gH) = \mu_x(g)$ for all $g \in G$ is a smooth G-equivariant diffeomorphism.*

In the above theorem, $G_x = \{g \in G \mid \mu(g, x) = x\}$ is the *isotropy subgroup* of G at $x \in M$. This reduces to an algebraic problem, by means of the following significant theorem ([2] Proposition 3.1):

Theorem 1.2. *Let G be a Lie group with connected subgroup H, and let G and H have Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively. There is a one-to-one correspondence between G-invariant Lorentz metrics on G/H, and inner products on $\mathfrak{g}/\mathfrak{h}$ that are* $\mathrm{ad}\mathfrak{h}$*-invariant (i.e., invariant under the action of $\mathfrak{h}$ by ad). (We do not require the inner products to be positive definite.)*

In the preceding theorem, $\mathrm{ad} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is given by the formula

$$\mathrm{ad}(X, Y) = [X, Y],$$

where $[\cdot, \cdot]$ denotes the Lie bracket on $\mathfrak{g}$. (See [4], Section 3.46 and Theorem 3.47 for details.)

By Theorem 1.1, to study homogeneous spaces, it is only necessary to consider Lie groups and their isotropy subgroups. There are generally many such isotropy subgroups, but the next theorem limits the number of quotient spaces that need to be considered.

Theorem 1.3. *Let G be a Lie group with closed subgroups H and K. If $\phi : G \to G$ is a smooth group isomorphism such that $\phi(H) = K$, then ϕ induces a G-equivariant diffeomorphism $\tilde{\phi} : G/H \to G/K$.*

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To summarize, *in order to find an invariant metric* γ *on a manifold* M *it is enough to find an invariant inner product* η *on a vector space* $\mathfrak{g}/\mathfrak{h}$. In this note, we will provide a detailed proof of this fact.

2. Review of Manifolds

This section is intended only as a brief review of manifolds and their properties. For further details see [1].

Definition 2.1. *A* topological manifold M *of dimension* n*, or* n-manifold*, is a topological space such that:*

(i) M *is Hausdorff,*

(ii) M *is locally Euclidean of dimension* n*, and*

(iii) M *has a countable basis of open sets.*

These requirements imply for each point in M*, there exists a* coordinate chart (U, ϕ)*, where* U *is an open set in* M *about the point and* ϕ *is a homeomorphism of* U *to a subset of* $\mathbb{R}^n$.

Definition 2.2. *Let* M *and* N *be manifolds. A function* $f : M \to N$ *is said to be* smooth *if it has continuous partial derivatives of all orders and each such derivative is independent of the order of differentiation. The set of all smooth real-valued functions from* M *to* N *is denoted by* $C^\infty(M, N)$*. In the special case* $N = \mathbb{R}$*, this is written simply* $C^\infty(M)$.

Definition 2.3. *Two coordinate charts* (U, ϕ) *and* (V, ψ) *are said to* C^∞-compatible *if both*

$$\phi \circ \psi^{-1} : \psi(U \cap V) \to \phi(U \cap V) \text{ and } \psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$$

are diffeomorphisms (i.e. C^∞ *maps with* C^∞ *inverses).*

Definition 2.4. *A* differentiable *or* C^∞-manifold *is a topological manifold* M *with a family of coordinate charts* $U = \{(U_\alpha, \phi_\alpha)\}$ *(called an* atlas*) such that*

(1) M *is covered by* $\{U_\alpha\}$*,*

(2) for any α *and* β *the charts* (U_α, ϕ_α) *and* (U_β, ϕ_β) *are* C^∞*-compatible, and*

(3) if a chart (V, ψ) *is* C^∞*-compatible with every* $(U_\alpha, \phi_\alpha) \in U$*, then* (V, ψ) *is in* U.

In order to study manifolds, it is necessary to define a few concepts.

Definition 2.5. *The* tangent space T_pM *to* M *at* p *is the set of all mappings* $X_p : C^\infty(M) \to \mathbb{R}$ *satisfying for* $\alpha, \beta \in \mathbb{R}$ *and* $f, g \in C^\infty(M)$*,* *(i)* $X_p(\alpha f + \beta g) = \alpha(X_p f) + \beta(X_p g)$ *(*linearity*), and*

(ii) $X_p(f \circ g) = (X_p f)(g(p)) + f(p)(X_p g)$ *(*Leibniz rule*).*

Any $X_p \in T_pM$ *is called a* tangent vector *at* p *in* M*.*

The tangent space at a point $p \in M$ is a vector space with operations defined as

$$(X_p + Y_p)f = X_pf + Y_pf \text{ and } (\alpha X_p)f = \alpha(X_pf),$$

for $X_p, Y_p \in T_pM$ with $f \in C^\infty(M)$ and $\alpha \in \mathbb{R}$.

Definition 2.6. *Let* M *and* N *be manifolds with* $F : M \to N$ *a smooth function. For* $p \in M$ *the* push-forward *of* F *at* p *is the smooth map*

$$F_* : T_pM \to T_{F(p)}N \textit{ defined by } F_*(X_p)f = X_p(f \circ F).$$

If $H = G \circ F$*, then the chain rule states that* $H_* = G_* \circ F_*$*.*

For a diffeomorphism of manifolds $F : M \to N$, if $\frac{\partial}{\partial x^1}|_p \cdots \frac{\partial}{\partial x^n}|_p$ is a basis for T_pM and $\frac{\partial}{\partial y^1}|_{F(p)} \cdots \frac{\partial}{\partial y^n}|_{F(p)}$ is a basis for $T_{F(p)}N$ then

$$(F_*)(\frac{\partial}{\partial x^i}) = \frac{\partial F^\alpha}{\partial x^i}|_p \frac{\partial}{\partial y^\alpha}. \tag{1}$$

Definition 2.7. *A* vector field X *on a manifold* M *is a function assigning to each* $p \in M$ *a vector* $X_p \in T_pM$ *such that* X_p *varies smoothly with respect to* p*.*

In order to make this more precise, it is necessary to give the following important definition.

Definition 2.8. *Let* M *be a manifold. Then define the* tangent bundle *to be*

$$TM = \bigcup_{p \in M} T_pM = \{(p, X_p) \mid p \in M, X_p \in T_pM\}.$$

The tangent bundle TM *is a smooth manifold of dimension* $2n$ *and comes equipped with the projection map* $\pi : TM \to M$ *where*

$$\pi(p, X_p) = p.$$

The set of all C^∞-vector fields on a manifold M is labeled $\mathfrak{X}(M)$.

A vector field then can be defined as a smooth map $X : M \to TM$ by $X(p) = (p, X_p)$, where $X_p \in T_pM$, such that $(\pi \circ X)(p) = p$.

Definition 2.9. *Let* M *and* N *be manifolds with* $F : M \to N$ *a smooth map. If* $X \in \mathfrak{X}(M)$ *and* $Y \in \mathfrak{X}(N)$ *then* X *and* Y *are said to be* F-related *if* $Y_{F(p)} = F_*(X_p)$ *for all* $p \in M$*. This is commonly written* $Y = F_*(X)$*.*

It can be shown that if $F : M \to N$ is a diffeomorphism, then each vector field X on M is F-related to a uniquely determined vector field Y on N.

Definition 2.10. *Let M be a manifold. If $F : M \to M$ is a diffeomorphism and $X \in \mathfrak{X}(M)$ such that $F_*(X) = X$ (i.e X is F-related to itself), then X is* invariant with respect to F *or* F-invariant.

Definition 2.11. *If M is a manifold with $p \in M$, then the* cotangent space *at p in M is the dual space T_p^*M. The dual space to T_pM is the set of all linear mappings $\omega_p : T_pM \to \mathbb{R}$. Any $\omega_p \in T_p^*M$ is called a* cotangent vector *at p in M.*

Given a basis $E_{1p}, \ldots, E_{np}$ of T_pM, there is a uniquely determined dual basis $\omega_p^1, \ldots, \omega_p^n$ satisfying $\omega_p^i(E_{jp}) = \delta_j^i$.

Definition 2.12. *Let M and N be manifolds with $F : M \to N$ a smooth function. For $p \in M$ the* pull-back *of F at p is the smooth map*

$$F^* : T_{F(p)}^*N \to T_p^*M$$

defined by

$$F^*(\omega_{F(p)})(X_p) = \omega_{F(p)}(F_*(X_p)).$$

Definition 2.13. *Let M be a manifold. Then define the* cotangent bundle *to be*

$$T^*M = \bigcup_{p \in M} T_p^*M = \{(p, \omega_p) \mid p \in M, \omega_p \in T_p^*M\}.$$

Definition 2.14. *A* one-form *ω on a manifold M is a smooth function $\omega : M \to T^*M$ assigning to each $p \in M$ a cotangent vector $\omega_p \in T_p^*M$.*

The map F_* does not always map vector fields on M to vector fields on N, but F^* does determine a one-form on M given a one-form on N.

Definition 2.15. *If V is a vector space over $\mathbb{R}$, then a* bilinear form *on V is defined to be a map $\Phi : V \times V \to \mathbb{R}$ that is linear in each variable separately.*

Assume the vector space V is n-dimensional. In this case, a basis $\{e_1, \ldots, e_n\}$ can be chosen for V. For a bilinear form Φ on V, let $\alpha_{ij} = \Phi(e_i, e_j)$ for $i = 1, \ldots, n$ and $j = 1, \ldots, n$. These n^2 values are called the *components* of Φ with respect to the basis $\{e_1, \ldots, e_n\}$ and completely determine the value of Φ (this is assured by the bilinearity of Φ). A matrix A can be defined to have components $A_j^i = \alpha_{ij}$, establishing a one-to-one correspondence between $n \times n$ matrices and bilinear forms on V with the given basis.

Definition 2.16. *Let Φ be a bilinear form on a vector space V. The bilinear form Φ is* symmetric *if $\Phi(v, w) = \Phi(w, v)$. It is* skew-symmetric *if $\Phi(v, w) = -\Phi(w, v)$. In addition, a symmetric bilinear form is called* positive definite *if $\Phi(v, v) \geq 0$ and $\Phi(v, v) = 0$ if and only if $v = 0$. A positive definite bilinear form on a vector space V is often called an* inner product *on V.*

Definition 2.17. *Let M be a manifold. A smooth function γ is a* field of bilinear forms *on M if it assigns to each point $p \in M$ a bilinear form γ_p on T_pM, so that $\gamma_p : T_pM \times T_pM \to \mathbb{R}$.*

As was the case for one-forms, if $F : M \to N$ is a smooth mapping of manifolds and γ is a field of bilinear forms on N then $F^*\gamma$ is a field of bilinear forms on M defined by the formula

$$(F^*\gamma)(X_p, Y_p) = \gamma(F_*(X_p), F_*(Y_p)).$$

Furthermore, if γ is symmetric then $F^*\gamma$ is symmetric; and if the field of symmetric bilinear forms γ is positive definite and F is an immersion (F^* is injective), then $F^*\gamma$ is symmetric and positive definite.

Definition 2.18. *Let M be a manifold with a field of symmetric, positive definite bilinear forms γ. Then M is called a* Riemannian manifold *and γ a* Riemannian metric*.*

If a bilinear form is positive definite, then the corresponding matrix has maximal index. So if the bilinear form γ is on an n-dimensional vector space, then the corresponding matrix has index n. This is not the only case of interest, however.

Definition 2.19. *Let M be an n-dimensional manifold with a field of symmetric bilinear forms γ. If at each point $p \in M$, the matrix corresponding to the bilinear form γ_p on T_pM has index $n-1$, then M is called a* Lorentz manifold *and γ a* Lorentz metric*.*

Definition 2.20. *Two Riemannian (Lorentz) manifolds M_1 and M_2 with Riemannian (Lorentz) metrics γ_1 and γ_2 are said to be* isometric *if there exists a diffeomorphism $F : M_1 \to M_2$ such that $F^*\gamma_2 = \gamma_1$. A map $F : M \to M$ is called an* isometry *of a metric γ if $F^*\gamma = \gamma$.*

3. Review of Group Actions

The following section discusses *action of a group on a set*. The group G will often be a Lie group and the set X will often be a manifold, but the discussion will be kept arbitrary for now.

Definition 3.1. *Let G be a group and X a set. A mapping $\mu : G \times X \to X$ is said to be a* group action *on the set if the following two conditions are satisfied:*

(1) If $e \in G$ is the identity element, then

$$\mu(e, x) = x \text{ for all } x \in X.$$

(2) If $g_1, g_2 \in G$, then

$$\mu(g_2, \mu(g_1, x)) = \mu(g_2 g_1, x) \text{ for all } x \in X.$$

The group G is said to act *on the set X by the action μ. For simplicity, $\mu(g,x)$ is often written gx.*

The action defined above is actually a *left* action of a group on a set, so called because the group element multiplies on the left side of the point. In order to define a *right* action, change property (2) as follows:

(2) If $g_1, g_2 \in G$, then

$$\mu(g_2, \mu(g_1, x)) = \mu(g_1 g_2, x) \text{ for all } x \in X.$$

If G is any group, then $\mu : G \times G \to G$ can be defined for $g_1, g_2 \in G$ by $\mu(g_1, g_2) = g_1 g_2$, the group multiplication. This defines an action of the group G on itself. Closure of the group assures the map is well-defined, condition (1) is satisfied by the identity axiom, and associativity confirms condition (2).

In many cases it is convenient to think of the action on a set by a fixed element of a group. Given a group G acting on a set X by an action μ as described above, if $g \in G$, then define $\mu_g : X \to X$ by

$$\mu_g(x) := \mu(g, x) \text{ for all } x \in X.$$

For any $g \in G$ the map μ_g has an inverse and $(\mu_g)^{-1} = \mu_{g^{-1}}$. Fix $g \in G$ and $x \in X$, then following from the definition of μ_g and $\mu_{g^{-1}}$,

$$\mu_{g^{-1}} \circ \mu_g(x) = \mu_{g^{-1}}(\mu(g, x)) = \mu(g^{-1}, \mu(g, x)).$$

The mapping μ is a group action, so condition (1) may be imposed, giving the following result:

$$\mu(g^{-1}, \mu(g, x)) = \mu(g^{-1} g, x) = \mu(e, x).$$

Condition (2) of a group action implies $\mu(e, x) = x$, hence the result:

$$\mu_{g^{-1}} \circ \mu_g(x) = x.$$

A similar argument shows $\mu_g \circ \mu_{g^{-1}}(x) = x$ and so ${\mu_g}^{-1} = \mu_{g^{-1}}$.

In addition, for any $x \in X$, μ can define another map $\mu_x : G \to X$ as follows: for any $g \in G$, $\mu_x(g) = \mu(g, x)$.

Definition 3.2. *Let a group G act on a set X by μ. The action μ is* effective *if μ_g being the identity implies $g = e$. In this case, G is said to act on X* effectively.

By the group axioms, there exists only one unique identity element in any group G, so clearly the only element in G which fixes all elements of G with respect to the group multiplication is the identity. Thus any group acts effectively on itself.

Definition 3.3. *Let a group G act on a set X by μ. Let $O_x = \{y \in X \mid y = gx$ for some $g \in G\}$. The set O_x is called the* orbit *of x in X. If y lies in the orbit of x, so that $y = gx$ for some $g \in G$, it is sometimes said that x can be moved to y by g. If $O_x = \{x\}$, i.e. $gx = x$ for all $g \in G$, x is called a* fixed point *of X.*

Definition 3.4. *Let a group G act on a set X by μ. The action μ is called a* transitive action *of G on X if for any $x, y \in X$ there exists a $g \in G$ such that $y = \mu(g, x)$. In this case, G is said to act* transitively *on X. If an action is transitive, any point can be moved to any other point, so the orbits are the entire set X.*

Note that any group G acts transitively on itself. If μ is defined by the group operation. Then for any $g_1, g_2 \in G$, there exists $g = g_2 g_1^{-1} \in G$ such that $g_2 = \mu(g, g_1)$. The existence of $g = g_2 g_1^{-1}$ is assured by the group axioms.

Definition 3.5. *Let a group G act on a set X by μ. Given $x \in X$, the* isotropy *of G at x is defined as*

$$G_x = \{g \in G \mid \mu(g, x) = x\}.$$

The subgroup G_x is often called the stabilizer *of x.*

Lemma 3.1. *Let G be a group acting on a set X. The isotropy G_x of G at $x \in X$ is a subgroup of G.*

Proof. If G_x is to be a subgroup of G, then it must satisfy the axioms of subgroups:

(1) Closure: Let $g_1, g_2 \in G_x$ for some $x \in X$. Then $g_1 x = x$ and $g_2 x = x$. It follows that

$$(g_1 g_2)x = g_1(g_2 x) = g_1 x = x,$$

so $g_1 g_2 \in G_x$.

(2) Identity: If $e \in G$ is the identity element, then by definition of group action, $ex = x$, so $e \in G_x$.

(3) Inverses: Let $g \in G_x$, so that $gx = x$. G is a group so there exists a $g^{-1} \in G$. If $x = gx$, then

$$g^{-1}x = g^{-1}(gx) = (g^{-1}g)x = x.$$

Therefore $g^{-1} \in G_x$.

The three axioms satisfied, G_x is a subgroup. □

The previous definitions for groups and group actions can be applied to the special case of a Lie group G acting on a manifold M.

Definition 3.6. *A group G is a* Lie group *if G is a manifold and both the group operation and the mapping of an element to its inverse are C^∞:*

(1) $\theta_1 : G \times G \to G$ defined by $\theta_1(g_1, g_2) = g_1 \cdot g_2$ and

(2) $\theta_2 : G \to G$ *defined by* $\theta_2(g) = g^{-1}$.

Definition 3.7. *Let* G *be a Lie group and* M *a manifold. A mapping* $\mu : G \times M \to M$ *is said to be a* Lie group action *on the manifold if* μ *is smooth (or* C^∞*) and satisfies the conditions for group action given in Definition 3.1.*

If a Lie group G acts on a manifold M by μ for $g \in G$, $\mu_g : M \to M$ defined for $x \in M$ by $\mu_g(x) = \mu(g, x)$ is a diffeomorphism. It is smooth because group action is smooth by assumption and it has an inverse which must also be smooth. The map $\mu_x : G \to M$ for any $x \in M$ defined by $\mu_x(g) = \mu(g, x)$ for all $g \in G$ is also smooth for the same reasons.

Theorem 3.2. *If* G *is a Lie group acting smoothly on a manifold* M*, then for any* $x \in M$ *the isotropy* G_x *is a closed subgroup of* G*.*

Proof. It has already been shown that the isotropy is a subgroup. It only remains to show that G_x is closed as a subset in G. By assumption, G acts on M by a smooth function, say μ. Consider any $x_0 \in M$. Now define a map $F : G \to M$ by

$$F(g) = \mu_{x_0}(g) = \mu(g, x_0)$$

for all $g \in G$. The map μ_{x_0} is smooth so F is smooth also. Next note that

$$F^{-1}(x_0) = \{g \in G \mid \mu_{x_0}(g) = x_0\},$$

but this is simply G_{x_0}. The singleton $\{x_0\}$ is closed in M, so by continuity of F, $G_{x_0} = F^{-1}(x_0)$ is closed in G. The choice of $x_0 \in G$ was completely arbitrary, of course, so G_x is closed in G for any $x \in M$. □

In order to reach the main goal of homogeneous spaces, it will be important to have one more definition from group theory.

Definition 3.8. *Let* H *be a subgroup of a group* G*. The* left coset *of* $g \in G$ *with respect to* H *is defined as the set* gH*.*

This notion of cosets can be use to define an equivalence relation $\sim$ as follows: two elements $g, \hat{g} \in G$ are said to be equivalent, or $g \sim \hat{g}$, if they are in the same coset, i.e. if $gH = \hat{g}H$. Another way of saying the same thing is that $\hat{g} \in gH$; but this means there exists an $h \in H$ such that $\hat{g} = gh$. It still remains to be shown, however, that $\sim$ satisfies the three axioms of an equivalence relation:

(i) Reflexivity: For all $g \in G$, $g \sim g$.

(ii) Symmetry: For all $g_1, g_2 \in G$, if $g_1 \sim g_2$ then $g_2 \sim g_1$.

(iii) Transitivity: For all $g_1, g_2, g_3 \in G$, if $g_1 \sim g_2$ and $g_2 \sim g_3$, then $g_1 \sim g_3$.

For (i), note that the identity element e in G must be in all subgroups of G, thus $e \in H$. From the definition of the identity element, if $g \in G$, then $g = ge \in gH$, so $g \sim g$. Now, for $g_1, g_2 \in G$, if $g_1 \sim g_2$ then $g_2 \in g_1H$, or $g_2 = g_1h$ for some $h \in H$. This implies that $g_2h^{-1} = g_1$. H is a subgroup of G, so if $h \in H$ then $h^{-1} \in H$

and $g_1 = g_2h^{-1} \in g_2H$, and so $g_2 \sim g_1$, satisfying (ii). Finally, if $g_1, g_2, g_3 \in G$ and $g_1 \sim g_2$ and $g_2 \sim g_3$, then there exist $h_1, h_2 \in H$ such that $g_2 = g_1h_1$ and $g_3 = g_2h_2$. Let $h_3 = h_1h_2$, which by closure is in H. Then

$$g_1h_3 = g_1(h_1h_2) = (g_1h_1)h_2 = g_2h_2 = g_3.$$

Thus there exists $h_3 \in H$ such that $g_3 = g_1h_3$, and so $g_1 \sim g_3$, proving (iii).

Definition 3.9. *Let H be a subgroup of G. Then G/H, the* quotient space *of H in G, is the set of all (left) cosets of G.*

Theorem 3.3. *Let G be a Lie group with subgroup H. Then G/H is Hausdorff with the induced quotient topology if H is a closed subspace of G.*

Proof. Before G/H can be called Hausdorff, a topology must be given. Given a group G with subgroup H, the *induced quotient topology* on G/H is obtained by defining a set $V \in G/H$ to be *open* in G/H if $\pi^{-1}(V)$ is an open set in G, where π is the *quotient projection* $\pi : G \to G/H$ defined by the formula $\pi(g) = gH$ for $g \in G$. It follows that π is continuous by definition. Additionally, π is an open map, i.e. if U is open in G, then $\pi(U)$ is open in G/H. In order to see this, it is enought to show that for U open in G, $\pi^{-1}(\pi(U))$ is also open in G, but

$$\pi^{-1}(\pi(U)) = \bigcup_{g \in G} gU,$$

is open because it is the union of open sets.

Now proceed by defining a subset $R \subset G \times G$ as follows:

$$R = \{(g_1, g_2) \mid g_1 \sim g_2\}.$$

Continue by defining a map $F : G \times G \to G$ by $F(g_1, g_2) = g_2^{-1}g_1$. This map is C^∞ because group multiplication is C^∞ for a Lie group. It follows that $F^{-1}(H) = R$. So by continuity of F, if H is closed in G, then R is closed in $G \times G$. Therefore, it is sufficient to show that R closed in $G \times G$ implies that G/H is Hausdorff. Assume $x, y \in G/H$ are nonidentical. The points x and y are in the quotient space, thus there exist two elements $g_1, g_2 \in G$ such that $x = \pi(g_1)$ and $y = \pi(g_2)$. By assumption $\pi(g_1) \neq \pi(g_2)$, so g_1 and g_2 are not in the same coset; hence $g_1 \nsim g_2$ and so $(g_1, g_2) \notin R$. The complement of R is open, thus there is an open set $U \times V$ in $G \times G$ containing (g_1, g_2) such that $U \times V \cap R = \emptyset$. Translated this means no elements in U are equivalent to any elements in V, or $\pi(u) \neq \pi(v)$ for all $u \in U$ and $v \in V$ implying $\pi(U) \cup \pi(V) = \emptyset$. Therefore there exist disjoint sets $\pi(U), \pi(V)$ in G/H such that $x \in \pi(U)$ and $y \in \pi(V)$; these sets are open because the quotient projection is an open map. The quotient G/H is Hausdorff, as required. □

The converse of this theorem is also true, but it is not needed in this paper.

If a group G acts on a set X, then the isotropy subgroup G_x for $x \in X$ is a subgroup of G. Thus G/G_x defines a quotient space. In the case of a Lie group, the isotropy subgroup G_x in G is closed, so G/G_x is Hausdorff.

Theorem 3.4. *Let G be a Lie group with subgroup H such that H is closed as a subset of G. Then*

1. *the quotient space G/H is a manifold of dimension* $\dim(G) - \dim(H)$*,*
2. *the projection map $\pi : G \to G/H$ is smooth, and*
3. *there exists a smooth local section $\sigma : G/H \to G$.*

PROOF. Recall that given a group G with subgroup H, the induced quotient topology was defined so that a set $V \in G/H$ is open if $\pi^{-1}(V)$ is open in G, where π is the quotient projection defined above. The map π is continuous by definition, but it should be noted that π^{-1} is the preimage mapping and not the inverse function. Indeed, the inverse of π need not exist. There may be more than one element of the group in the same coset and the preimage would map the image back to both of these. Locally, however, a smooth function $\sigma : G/H \to G$ can be defined so that

$$(\pi \circ \sigma)(x) = x \tag{2}$$

for any $x \in G/H$. Continuing will require a lemma, given here without proof, which can be found in [1], pages 169-170.

Lemma 3.5. *Let G be an n-dimensional Lie group and H be an m-dimensional Lie subgroup of G which is closed as a subset. For any $\epsilon > 0$ there exists a cubical neighborhood or chart (U, ϕ) of any $g \in G$ such that if (U, ϕ) on G has local coordinates $x^1, \ldots, x^n$ then*

(i) $\phi(p) = (0, \ldots, 0)$,

(ii) $\phi(U) = C^n_\epsilon(0)$*, and*

(iii) $\phi(U \cap H) = \{x \in C^n_\epsilon(0) \mid x^{m+1} = \cdots = x^n = 0\}$.

Now it can be shown that σ is locally smooth. Given G is a manifold, there exists at any point an open neighborhood U and a diffeomorphism $\phi : U \to \phi(U) \subseteq \mathbb{R}^n$, where n is the dimension of the Lie group G. By assumption a Lie subgroup H has been given, say of dimension m. Then H can be given the manifold structure described in the preceding lemma. Define a new map $\tilde{\phi}$ such that it's inverse is $\tilde{\phi}^{-1} = \pi \circ \phi^{-1} \mid_S : S \to \pi(U)$ where $S = \phi(U \cup H)$. So $\tilde{\phi} : \pi(U) \to S$. Next, if $L_a : G \to G$ is left translation by some $a \in G$ so that $L_a(g) = ag$ for all $g \in G$, define $\tilde{L}_a : G/H \to G/H$ to be the induced left translation on G/H defined by

$$\tilde{L}_a(gH) = (ag)H$$

for any $g \in G$. Smoothness of $\tilde{L}_a$ follows from the smoothness of L_a. Finally define $\sigma : G/H \to G$ as

$$\sigma = L_a \circ \phi^{-1} \circ \tilde{\phi} \circ \tilde{L}_{a^{-1}}.$$

Therefore σ is smooth as the composition of smooth functions. Of course, note that these functions were defined only on neighborhoods, and so σ is defined only locally. While σ is not exactly an inverse of π, it will suffice in many cases. For fuller discussion, see [4], pp. 120-12l.

Now that the section σ has been defined, it can be used to determine a chart on G/H. Let (U, ϕ) be a chart on G with local coordinates $x^1, \ldots, x^m, x^{m+1}, \ldots, x^n$. Let

$$A \ = \ \{ \ q \in U \mid x^1(q) \ = \ \cdots \ = \ x^m(q) \ = \ 0 \ \}$$

and define

$$\psi : A \to C_\epsilon^{n-m}(0) \ \text{ by } \ \psi(q) = (x^{m+1}(q), \ldots, x^n(q)).$$

By the choice of U, $A \cup H$ can contain only one point. This also holds for the intersection of A with each coset of H, so $\pi(A)$ is open in G/H. Define a chart on G/H to be $(\pi(A), \psi \circ \sigma)$. This chart is differentiable, hence G/H is a manifold. Additionally,

$$\psi \circ \sigma : \pi(A) \to C_\epsilon^{n-m}(0)$$

is a diffeomorphism, so $\dim(G/H) = n - m = \dim(G) - \dim(H)$. $\square$

For a group G with subgroup H, a natural action of the group G can be defined on the quotient G/H.

Theorem 3.6. *Let G be a Lie group with closed subgroup H. If $\mu : G \times G/H \to G/H$ is defined by*

$$\mu(\tilde{g}, gH) = (\tilde{g}g)H,$$

then μ is a C^∞ group action of G on G/H. The isotropy subgroup at $x = gH$ is

$$G_x = \{\hat{g} \in G \quad | \quad (\hat{g}g)H = gH\}.$$

Proof.If this is to be a group action, the two axioms must be satisfied.

(1) If $e \in G$ is the identity element and gH is the left coset of $g \in G$, then

$$\mu(e, gH) = (eg)H = gH$$

and

(2) If g_1 and g_2 are in G and gH is the left coset of $g \in G$, then

$$\mu(g_2, \mu(g_1, gH)) = \mu(g_2, (g_1g)H) = (g_2(g_1g))H.$$

G is a group, so associativity of group multiplication assures $(g_2(g_1g)) = ((g_2g_1)g)$. Therefore

$$\mu(g_2, \mu(g_1, gH)) = ((g_2g_1)g)H = \mu(g_2g_1, gH).$$

Therefore μ is a group action and it only remains to show that the action is smooth.

Let U be any open set in G/H. Define $W \subseteq \ G \times \ G$ such that if $(g_1, g_2) \in \ W$, then

$$g_1 \cdot g_2 \in \pi^{-1}(U),$$

which is open in G by continuity of π. The group multiplication is continuous so, using the map given in Definition 3.6,

$$W = \theta_1^{-1}(\pi^{-1}(U))$$

is open in $G \times G$. Now let the induced open map

$$\tilde{\pi} : G \times G \to G \times G/H$$

be defined by the formula

$$\tilde{\pi}(g_1, g_2) = (g_1, \pi(g_2)).$$

It follows that $\tilde{\pi}(W) = \mu^{-1}(U)$ because W was made up of all elements g_1 and g_2 in G such that $\pi(g_1 g_2) \in U$ or $(g_1 g_2)H \in U$, hence $\tilde{\pi}(W)$ contains every pair of elements $(g_1, g_2 H) \in G \times G/H$ such that $\mu(g_1, g_2 H) \in U$. Therefore U open in G/H implies $\mu^{-1}(U)$ is open in $G \times G/H$ and so μ is a continuous action. □

Definition 3.10. *A manifold M is said to be a* homogeneous space *of a Lie group G if there is a transitive C^∞ action of G on M.*

In the next section it is shown that every homogeneous space is equivalent to a quotient G/H.

4. G-Equivariant Diffeomorphisms

It has been shown that homogeneous spaces can be constructed using Lie groups and their subgroups. One might wonder how many homogeneous spaces can be constructed from a given Lie group. A homogeneous space is a manifold with a transitive group action; so for two homogeneous spaces to be equivalent, not only must they be equivalent as manifolds (i.e diffeomorphic), but the group actions must be equivalent as well.

Definition 4.1. *Let*

$$\mu_1 : G \times M_1 \to M_1 \text{ and } \mu_2 : G \times M_2 \to M_2$$

be actions of a Lie group on the manifolds M_1 and M_2. Then the actions μ_1 and μ_2 are equivalent *if there exists a diffeomorphism $F : M_1 \to M_2$ such that*

$$F(\mu_1(g, x)) = \mu_2(g, F(x))$$

for all $g \in G$ and $x \in M_1$. In this case the diffeomorphism F is called G-equivariant.

Theorem 4.1. *Let G be a Lie group acting transitively on a manifold M by a smooth action μ. If x is any point in M, then the mapping $F : G/G_x \to M$ defined by $F(gH) = \mu_x(g)$ is a smooth G-equivariant diffeomorphism. In particular, $M \cong G/G_x$.*

Proof. Begin by letting $H = G_{x_0}$ for any fixed $x_0 \in M$. Define $F : G/H \to M$ by

$$F(gH) = \mu_{x_0}(g) = \mu(g, x_0) \tag{3}$$

for any coset $gH \in G/H$. In order for this map is well-defined, for any $g, \tilde{g} \in G$ such that $gH = \tilde{g}H$ it must be shown that $F(gH) = F(\tilde{g}H)$. If $gH = \tilde{g}H$, then $g^{-1}\tilde{g} \in H = G_{x_0}$, so

$$x_0 = \mu(g^{-1}\tilde{g}, x_0).$$

From this equation, using the rules of group action, it follows that

$$\begin{aligned}\mu(g,x_0) &= \mu\big(g,\mu(g^{-1}\tilde{g},x_0)\big) = \mu\big(g(g^{-1}\tilde{g}),x_0\big)\\ &= \mu\big((gg^{-1})\tilde{g},x_0\big) = \mu(\tilde{g},x_0).\end{aligned}$$

Therefore, $F(gH) = F(\tilde{g}H)$, as required.

In order for F to be a bijection, it must be both onto and one-to-one. The action μ is transitive, so for any $x \in M$ there exists a $g \in G$ such that $x = \mu(g,x_0) = \mu_{x_0}(g) = F(gH)$; hence F is onto.

Now let $F(g_1H) = F(g_2H)$ for any $g_1, g_2 \in G$. If this implies that $g_1H = g_2H$, then F is one-to-one. If $F(g_1H) = F(g_2H)$ then $\mu_{x_0}(g_1) = \mu_{x_0}(g_2)$. Furthermore, this shows that $g_1x_0 = g_2x_0$; which implies

$$g_1^{-1}(g_1x_0) = g_1^{-1}(g_2x_0). \tag{4}$$

Now applying the rules of group action,

$$\begin{aligned}&g_1^{-1}(g_1x_0) = (g_1^{-1}g_1)x_0 = x_0, \quad \text{and}\\ &g_1^{-1}(g_2x_0) = (g_1^{-1}g_2)x_0.\end{aligned}$$

Substituting these values into Equation 4,

$$x_0 = (g_1^{-1}g_2)x_0.$$

Therefore, $g_1^{-1}g_2 \in G_{x_0} = H$. This amounts to saying that g_1 and g_2 are in the same coset, or $g_1H = g_2H$, as required. It follows that F is a one-to-one mapping.

The map F is a bijection, but in order to be a diffeomorphism, it must be C^∞ as well. This becomes clear, however, after considering the locally smooth section map $\sigma : U \to G$, for all $U \subseteq G/H$, defined in Equation 2 of Section 3 by the identity

$$(\pi \circ \sigma)(x) = x$$

for all $x \in U$. Any section is locally smooth and so for every $x \in G/H$ there exists a section σ_x on a neighborhood U_x such that

$$F(x) = \mu_{x_0} \circ L_e \circ \sigma_x(x),$$

where L_e is left multiplication by the identity element in G. Such an equation exists for any $x \in G/H$. Thus F is the composition of smooth maps; hence, smooth itself.

Now it just remains to show that F is G-equivariant. The action of G on M has already been defined by μ, so let the natural action of G on G/H (as defined in Theorem 3.6) be denoted by $\hat{\mu}$. It must be shown that if $g \in G$ and $x \in G/H$, then

$$F\big(\hat{\mu}(g,x)\big) = \mu\big(g,F(x)\big). \tag{5}$$

If $x \in G/H$, then $x = \tilde{g}H$ for some $\tilde{g} \in G$. It follows from Equation 3 that

$$\begin{aligned}F\big(\hat{\mu}(g,x)\big) &= F\big(\hat{\mu}(g,\tilde{g}H)\big) = F\big((g\tilde{g})H\big) = \mu(g\tilde{g},a)\\ &= \mu\big(g,\mu(\tilde{g},a)\big) = \mu\big(g,F(\tilde{g}H)\big) = \mu\big(g,F(x)\big).\end{aligned}$$

Therefore Equation 5 is satisfied, as required. □

Corollary 4.2. *Let G be a Lie group which acts transitively on a manifold M by a smooth action μ. If $H = G_x$ for any $x \in M$, then*
$$T_xM \cong \frac{T_eG}{T_eH}.$$

Before this can be proved, a lemma is needed.

Lemma 4.3. *Let G be a Lie group which acts transitively on a manifold M by a smooth action μ. If $H = G_x$ for any $x \in M$ and if $\mu_x : G \to M$ is defined $\mu_x(g) = \mu(g, x)$ for all $g \in G$, then $(\mu_x)_* : T_eG \to T_xM$, and*
$$\text{image}(\mu_{x*}) = T_xM \text{ and } \ker(\mu_{x*}) = T_eH.$$

Proof. The map μ_x is from G to M, so by definition $(\mu_x)_* : T_eG \to T_{\mu_x(e)}M$. Part (1) of Definition 3.1 implies $\mu_x(e) = \mu(e, x) = x$, so $T_{\mu_x(e)}M = T_xM$. This proves the first claim.

In order to show $\text{im}(\mu_{x*}) = T_xM$, consider the tangent vector $v \in T_xM$. There exists some integral curve $\gamma(t) \in M$ such that $\dot\gamma(0) = v$. The action μ is transitive, so μ_x maps onto M. It follows that there is some curve $\gamma_1(t) \in G$ such that $\gamma(t) = \mu_x\big(\gamma_1(t)\big)$. Let $X_e = \dot\gamma_1(0) \in T_eG$. Then
$$(\mu_x)_*(X_e) = (\mu_x)_*\big(\dot\gamma_1(0)\big) = \dot\gamma(0) = v$$
and so $(\mu_x)_*$ is onto. Therefore $\text{image}(\mu_{x*}) = T_xM$.

It remains to show that $\ker(\mu_{x*}) = T_eH$. Let $X_e \in T_eH$; there exists an integral curve $\gamma(t) \in H$ such that $X_e = \dot\gamma(0)$. Calculating the image of this curve under μ_x will give an integral curve of $(\mu_x)_*X_e$ in M:
$$\mu_x\big(\gamma(t)\big) = \mu\big(\gamma(t), x\big) = x,$$
because $\gamma(t) \in H = G_x$ for each t. Therefore,
$$(\mu_x)_*X_e = \frac{d}{dt}\mu_x\big(\gamma(t)\big) \mid_{t=0} = \frac{d}{dt}(x) \mid_{t=0} = 0.$$
Therefore, $(\mu_x)_*X_e = 0$ and so $X_e \in \ker(\mu_{x*})$ for any $X_e \in T_eH$, implying
$$T_eH \subseteq \ker(\mu_{x*}).$$

For the map $(\mu_x)_* : T_eG \to T_xM$,
$$\dim\big(\ker(\mu_{x*})\big) + \dim\big(\text{im}(\mu_{x*})\big) = \dim(T_eG). \tag{6}$$
Using this equation, remembering that $\dim(G) = \dim(T_eG)$ and $\dim(M) = \dim(T_xM)$,
$$\begin{aligned}\dim\big(\ker(\mu_{x*})\big) &= \dim(T_eG) - \dim\big(\text{im}(\mu_{x*})\big)\\ &= \dim(T_eG) - \dim(T_xM)\\ &= \dim(G) - \dim(M).\end{aligned}$$
From Theorem 4.1, $M \cong G/H$, so
$$\begin{aligned}\dim\big(\ker(\mu_{x*})\big) &= \dim(G) - \dim(M) = \dim(G) - \dim(G/H)\\ &= \dim(G) - \dim(G) + \dim(H) = \dim(H) = \dim(T_eH).\end{aligned}$$

Thus $\dim\big(\ker(\mu_{x*})\big) = \dim(T_eH)$; but $T_eH \subseteq \ker(\mu_{x*})$, so

$$T_eH = \ker(\mu_{x*}),$$

as required. Hence, the proof is complete. $\square$

PROOF OF COROLLARY. The map $\mu_{x*} : T_eG \to T_xM$ is a vector space homomorphism, so it must be true that

$$\frac{T_eG}{\ker(\mu_{x*})} \cong \mathrm{im}(\mu_{x*}).$$

The lemma then implies that

$$\frac{T_eG}{T_eH} \cong T_xM, \tag{7}$$

as required. $\square$

Theorem 4.4. *Let G be a Lie group with closed subgroups H and K. If $\phi : G \to G$ is a smooth group isomorphism such that $\phi(H) = K$, then ϕ induces a G-equivariant diffeomorphism $\tilde{\phi} : G/H \ \to \ G/K$.*

PROOF. Define the induced mapping $\tilde{\phi} : G/H \to G/K$ by

$$\tilde{\phi}(gH) = \phi(g)K.$$

In order to show this mapping is well-defined, it is necessary to show $\tilde{\phi}(gH) = \tilde{\phi}(\hat{g}H)$ for any two $g, \hat{g} \in G$ in the same coset. Assume $gH = \hat{g}H$. Then it must be that $g^{-1}\hat{g} \in H$. By assumption $\phi(H) = K$, so it follows that $\phi(g^{-1}\hat{g}) \in K$. However, ϕ is a group isomorphism and so $\phi(g^{-1}\hat{g}) = \phi(g)^{-1} \cdot \phi(\hat{g})$. Thus $\phi(g)K = \phi(\hat{g})K$, implying $\tilde{\phi}(gH) = \tilde{\phi}(\hat{g}H)$, as required.

To show $\tilde{\phi}$ is G-equivariant, first define two group actions:

$$\begin{aligned} &i. \quad \mu_1 : G \times G/H \to G/H, \ \text{ by } \ \mu_1(g_1, g_2H) = (g_1g_2)H, \ \text{ and} \\ &ii. \quad \mu_2 : G \times G/K \to G/K, \ \text{ by } \ \mu_1(g_1, g_2K) = (g_1g_2)K. \end{aligned}$$

It has already been shown that these are actions, so the proof will be omitted here. Now let $g \in G$ and $x \in G/H$. The goal is to show

$$\tilde{\phi}(\mu_1(g, x)) = \mu_2(\phi(g), \tilde{\phi}(x)). \tag{8}$$

If $x \in G/H$, then $x = \hat{g}H$ for some $\hat{g} \in G$. It follows from the definitions of the various mappings that

$$\tilde{\phi}(\mu_1(g, x)) = \tilde{\phi}(\mu_1(g, \hat{g}H)) = \tilde{\phi}((g\hat{g})H) = \phi(g\hat{g})K.$$

The map ϕ is an isomorphism, so

$$\phi(g\hat{g})K = (\phi(g)\phi(\hat{g}))K = \mu_2(\phi(g), \phi(\hat{g})K) = \mu_2(\phi(g), \tilde{\phi}(\hat{g}H)).$$

By definition $\hat{g}H = x$, so Equation 8 is satisfied. Thus $\tilde{\phi}$ is G-equivariant.

To show $\tilde{\phi}$ is one-to-one (or injective) assume $\tilde{\phi}(g_1H) = \tilde{\phi}(g_2H)$ for any $g_1, g_2 \in G$. If it can be shown that $g_1H = g_2H$ then $\tilde{\phi}$ is injective. If $\tilde{\phi}(g_1H) = \tilde{\phi}(g_2H)$, then using the definition of $\tilde{\phi}$, it follows that

$$\phi(g_1)K = \phi(g_2)K.$$

This implies that $\phi(g_1)^{-1}\phi(g_2) \in K$. As ϕ is an isomorphism $\phi(g_1)^{-1}\phi(g_2) = \phi(g_1^{-1}g_2) \in K$. This in turn implies that $g_1^{-1}g_2 \in H$, or $g_1H = g_2H$, as required.

If $\tilde{\phi}$ is an isomorphism, it must also be onto (or surjective). Let $y \in G/K$. Then there exists some $\hat{g} \in G$ such that $x = \hat{g}K$. The map ϕ was by assumption an isomorphism. Thus there is a $g \in G$ such that $\hat{g} = \phi(g)$. It follows that if $x = gH \in G/H$, then

$$\tilde{\phi}(x) = \tilde{\phi}(gH) = \phi(g)K = \hat{g}K = y.$$

Thus $\tilde{\phi}$ is surjective and so an isomorphism.

It only remains to show that $\tilde{\phi}$ is smooth. By a similar argument to that used earlier in the proof of Theorem 4.1,

$$\tilde{\phi} = \pi_2 \circ \phi \circ \sigma$$

and so $\tilde{\phi}$ is smooth as the composition of smooth maps. □

5. Linear Isotropy Representation

It is sometimes possible to think of a group in the more familiar setting of linear algebra.

Definition 5.1. *A* representation *of a group G on a vector space V is a homomorphism $\rho : G \to Gl(V)$, where $Gl(V)$ is the group of automorphisms of V. A representation ρ is said to be* faithful *if*

$$\ker(\rho) = \rho^{-1}(I) = e,$$

where I is the identity matrix in $Gl(V)$ and e is the identity element in the group G, i.e. the kernel *of ρ is trivial.*

Let a Lie group G act on a manifold M by the action μ. It has already been shown that for any $g \in G$ a diffeomorphism $\mu_g : M \to M$ can be defined by $\mu_g(x) = \mu(g, x) = gx$. Given a point $p \in M$, this induces another map $(\mu_g)_* : T_pM \to T_{gp}M$, as was defined in Section 2, Definition 2.6.

Now consider the special case when $g \in G_p$, the isotropy subgroup of G at $p \in M$. Then $gp = p$ so $(\mu_g)_* : T_pM \to T_pM$. Now define a new mapping $\rho : G_p \to Gl(T_pM)$ by

$$\rho(g) = (\mu_g)_*. \tag{9}$$

This defines a representation of the isotropy subgroup of p, called the *linear isotropy representation* of G_p. To show this claim is true, consider $g_1, g_2 \in G_p$. By definition, $\rho(g_1g_1) = (\mu_{g_1g_2})_*$. The rules of group action assure that $\mu_{g_1g_2} = \mu_{g_1}\mu_{g_2}$ and so, by Definition 2.6,

$$(\mu_{g_1g_2})_* = (\mu_{g_1})_*(\mu_{g_2})_*.$$

Furthermore, $(\mu_{g_1})_*(\mu_{g_2})_* = \rho(g_1)\rho(g_2)$. Therefore $\rho(g_1g_2) = \rho(g_1)\rho(g_2)$, as required.

For any $g \in G$, the map $\mu_g : M \to M$ is a diffeomorphism of manifolds. Applying Equation 1 to μ_g, if $\frac{\partial}{\partial x^1}|_p \cdots \frac{\partial}{\partial x^n}|_p$ is a basis for T_pM and $\frac{\partial}{\partial y^1}|_{\mu_g(p)} \cdots \frac{\partial}{\partial y^n}|_{\mu_g(p)}$ is a basis for $T_{\mu_g(p)}N$ then

$$(\mu_g)_*(\frac{\partial}{\partial x^i}) = \frac{\partial(\mu_g)^\alpha}{\partial x^i}|_p \frac{\partial}{\partial y^\alpha}.$$

Thus, for any $g \in G_p$, $\rho(g) = \frac{\partial \mu_g^i}{\partial x^j}(g,p)$.

Let γ be a metric on the manifold M. Recall that a map $\phi : M \to M$ is an *isometry* of γ if $\gamma_p = \phi^* \gamma_{\phi(p)}$. Given an action μ of G on M, if μ_g is an isometry of γ for all $g \in G$, then the group G is said to *act by isometries* with respect to the metric γ on the manifold M. When a Lie group acts on a manifold by isometries,

$$\gamma_p(X_p, Y_p) = (\mu_g^* \gamma)(X_p, Y_p)$$

for $X_p, Y_p \in T_pM$; however, a metric γ is a field of bilinear forms on M, so

$$(\mu_g^* \gamma_{\mu_g(p)})(X_p, Y_p) = \gamma_{\mu_g(p)}\Big((\mu_g)_* X_p, (\mu_g)_* Y_p\Big).$$

This gives the following equation:

$$\gamma_p(X_p, Y_p) = \gamma_{gp}\Big((\mu_g)_* X_p, (\mu_g)_* Y_p\Big), \tag{10}$$

which leads to a useful definition.

Definition 5.2. *Let a Lie group G act on a manifold M by the action μ. A metric γ on M is called G-*invariant *if Equation 10 holds for all $p \in M$ and $g \in G$. This is equivalent to saying G acts by isometries with respect to the metric γ on M.*

Return to the special case of $g \in G_p$. If g is in the isotropy subgroup of $p \in M$, then Equation 10 becomes, using Equation 9,

$$\gamma_p(X_p, Y_p) = \gamma_p\Big(\rho(g)X_p, \rho(g)Y_p\Big).$$

The mapping ρ is into $Gl(T_pM)$, so $\rho(g)$ is a nonsingular matrix preserving the metric γ. Therefore $\rho(g) \in O(r,s)$ where γ is of type (r,s) at $p \in M$. This can be formalized into the following theorem:

Theorem 5.1. *Let G be a Lie group acting on a manifold M by the action μ. If γ is a G-invariant metric on M, then γ_p is an inner product on T_pM which is invariant under the linear isotropy representation $\rho : G_p \to Gl(T_pM)$.*

6. Fundamental Theorem of Invariant Metrics on Homogeneous Spaces

Let a Lie group G act on a manifold M by the transitive action μ. It was shown in Theorem 5.1 that if γ is a G-invariant metric on M, then γ_p is an inner product on T_pM which is invariant under $\rho : G_p \to Gl(T_pM)$. In the case of a transitive action, the converse is also true.

Theorem 6.1. *Let a Lie group G act on a manifold M by the transitive action μ. If η is an inner product on T_pM which is invariant under the linear isotropy representation $\rho : G_p \to Gl(T_pM)$, then there exists a unique G-invariant metric γ on M such that $\gamma_p = \eta$.*

PROOF. Assume η is an inner product on T_pM which is invariant under the linear isotropy representation $\rho : G_p \to Gl(T_pM)$. Then

$$\mu_g^*\eta = \eta \text{ for all } g \in G_p.$$

Begin by defining γ pointwise. For any $q \in M$ there exists an $a \in G$ such that $q = \mu(a,p)$ by the transitivity of the action. Thus a metric at q can be defined

$$\gamma_q(X_q, Y_q) = (\mu_{a^{-1}}^*\eta)(X_q, Y_q) = \eta((\mu_{a^{-1}})_*X_q, (\mu_{a^{-1}})_*Y_q).$$

This is simply the pull-back of η from T_pM to $T_{ap}M$ by the diffeomorphism $\mu_{a^{-1}}$.

It must now be shown that γ is well-defined. There may be more than one $g \in G$ such that $\mu(g,p) = q$; i.e. there may exist $b \in G$ such that $a \neq b$ and $\mu(a,p) = \mu(b,p)$. The definition of γ_q should be independent of this choice, so

$$\mu_{a^{-1}}^*\eta = \mu_{b^{-1}}^*\eta.$$

If $\mu(a,p) = \mu(b,p)$, then $p = \mu(b^{-1}, \mu(a,p)) = \mu(b^{-1}a, p)$. Therefore, $b^{-1}a \in G_p$. Let $c = b^{-1}a \in G_p$. It follows that $b^{-1} = (b^{-1}a)a^{-1} = ca^{-1}$. Using this result,

$$\mu_{b^{-1}}^*\eta(X_q, Y_q) = \eta((\mu_{b^{-1}})_*X_q, (\mu_{b^{-1}})_*Y_q) = \eta((\mu_{ca^{-1}})_*X_q, (\mu_{ca^{-1}})_*Y_q).$$

Multiplication in a Lie group is smooth and $c \in G_p$ implies $\mu_c^*\eta = \eta$, so

$$\begin{aligned} \eta((\mu_{ca^{-1}})_*X_q, (\mu_{ca^{-1}})_*Y_q) &= \eta((\mu_c)_*(\mu_{a^{-1}})_*X_q, (\mu_c)_*(\mu_{a^{-1}})_*Y_q) \\ &= \mu_c^*\eta((\mu_{a^{-1}})_*X_q, (\mu_{a^{-1}})_*Y_q) \\ &= \eta((\mu_{a^{-1}})_*X_q, (\mu_{a^{-1}})_*Y_q) \\ &= \mu_{a^{-1}}^*\eta(X_q, Y_q). \end{aligned}$$

Combining these results: $\mu_{b^{-1}}^*\eta(X_q, Y_q) = \mu_{a^{-1}}^*\eta(X_q, Y_q)$, as required. Thus γ_q is well-defined for all points $q \in M$, hence γ is also well-defined.

It only remains to show that γ is G-invariant, or $\mu_g^*\gamma = \gamma$ for all $g \in G$. Consider any $q \in M$. The goal is to show for $g \in G$ that $\mu_g^*\gamma_q = \gamma_{g^{-1}q}$. The action is transitive, so there exists an $a \in G$ such that $q = \mu(a,p)$ (as before). If X, Y are vector fields on M and $g^{-1}q = r$, then

$$\begin{aligned} \mu_g^*\gamma_q(X_{g^{-1}q}, Y_{g^{-1}q}) &= \mu_g^*[\gamma_q(X_r, Y_r)] = \mu_g^*[(\mu_{a^{-1}}^*\eta)(X_r, Y_r)] \\ &= (\mu_g^*\mu_{a^{-1}}^*)[\eta(X_r, Y_r)] = (\mu_{a^{-1}g}^*)[\eta(X_r, Y_r)] \\ &= (\mu_{a^{-1}g}^*\eta)(X_r, Y_r) = (\gamma_{(g^{-1}a)p})(X_r, Y_r) \\ &= \gamma_{g^{-1}q}(X_{g^{-1}q}, Y_{g^{-1}q}). \end{aligned}$$

Therefore γ is G-invariant, as required.

It only remains to show uniqueness of the metric γ. Let $\tilde{\gamma}$ be another G-invariant metric on M such that $\tilde{\gamma}_p = \eta$. Again let q be any point in M. The action μ is transitive, so there exists a $g \in G$ such that $q = \mu(g,p) = \mu_g(p)$. The metrics γ and $\tilde{\gamma}$ on M are G-invariant, so

$$\gamma_p(X_p, Y_p) = \gamma_{\mu_g(p)}((\mu_g)_* X_p, (\mu_g)_* Y_p) = \gamma_q(\bar{X}_q, \bar{Y}_q)$$

and

$$\tilde{\gamma}_p(X_p, Y_p) = \tilde{\gamma}_{\mu_g(p)}((\mu_g)_* X_p, (\mu_g)_* Y_p) = \tilde{\gamma}_q(\bar{X}_q, \bar{Y}_q).$$

The metrics are equal at p, so by substitution, $\gamma_q(\bar{X}_q, \bar{Y}_q) = \tilde{\gamma}_q(\bar{X}_q, \bar{Y}_q)$. This is true for any $q \in M$ so $\gamma = \tilde{\gamma}$. □

Thus on a homogeneous space M there is a one-to-one correspondence between the G-invariant metrics on M and the inner products on T_pM invariant under the linear isotropy representation.

This section will be concluded with two theorems which will be given without proof. For further details [3].

Theorem 6.2. *Let G be a Lie group and M a manifold with a metric γ. If γ is a G-invariant metric, then the linear isotropy representation $\rho : G_p \to Gl(T_pM)$ is a faithful representation for any $p \in M$.*

Theorem 6.3. *Let G be a Lie group and M a manifold with a metric γ. If γ is a G-invariant metric, then $\dim(G) \leq \frac{1}{2}n(n+1)$, where n is the dimension of the manifold M.* [1]

7. Invariant Inner Products on $\mathfrak{g}/\mathfrak{h}$

Let a Lie group G act transitively on a manifold M by μ and let $H = G_{x_0}$ for some fixed $x_0 \in M$. In the previous section it was shown that there is a one-to-correspondence between the G-invariant metrics on M and inner products on $T_{x_0}M$ invariant under the linear isotropy representation, often called *ρH-invariant inner products* on $T_{x_0}M$. The goal now is to turn the problem of finding invariant inner products on a tangent space to a manifold into an algebraic problem. In this section the action of H on $T_{x_0}M$ (by ρ) will be shown to be equivalent to an action of H on the quotient space $\mathfrak{g}/\mathfrak{h}$, where $\mathfrak{g}$ and $\mathfrak{h}$ are the Lie algebras of G and H, respectively.

As G/H was the set of cosets gH for $g \in G$, $\mathfrak{g}/\mathfrak{h}$ is the set of cosets $X + \mathfrak{h}$ for $X \in \mathfrak{g}$. Similarly to the case of groups, $X + \mathfrak{h} = \{X + Y \mid Y \in \mathfrak{h}\}$, where $+$ is vector space addition. As before, $X_1 + \mathfrak{h} = X_2 + \mathfrak{h}$ if and only if $X_1 - X_2 \in \mathfrak{h}$.

Before the action of H on $T_{x_0}M$ can be replaced by an action of H on $\mathfrak{g}/\mathfrak{h}$, an action of H on $\mathfrak{g}/\mathfrak{h}$ must be defined. Begin by defining for $g \in G$ the map $\mathrm{Ad}(g) : \mathfrak{g} \to \mathfrak{g}$ by

$$\mathrm{Ad}(g)(X) = (L_g)_*(X)$$

for all $X \in \mathfrak{g}$, recalling $L_g : G \to G$ is left multiplication by $g \in G$.

[1] If $\dim(G) = \frac{1}{2}n(n+1)$, then M is a space of constant curvature.

Proposition 7.1. *If G is a Lie group, then for all $g \in G$ and $X \in \mathfrak{g}$,*

$$g \cdot X = \mathrm{Ad}(g)(X)$$

defines an action of G on $\mathfrak{g}$.

Proof. If this is to be a group action, it must satisfy the conditions of Definition 3.1. First, if $e \in G$ is the identity element, then for all $X \in \mathfrak{g}$,

$$e \cdot X = \mathrm{Ad}(e)(X) = (L_e)_*(X) = X,$$

as required. Secondly, let $g_1, g_2 \in G$ and $X \in \mathfrak{g}$. Then a simple, albeit long, calculation shows

$$\begin{aligned} g_2 \cdot (g_1 \cdot X) &= g_2 \cdot \mathrm{Ad}(g_1)(X) = \mathrm{Ad}(g_2)\big(\mathrm{Ad}(g_1)(X)\big) \\ &= \mathrm{Ad}(g_2)\big((L_{g_1})_*(X)\big) = (L_{g_2})_*\big((L_{g_1})_*(X)\big) \\ &= \big((L_{g_2})_*(L_{g_1})_*\big)(X) = (L_{g_2} \circ L_{g_1})_*(X) \\ &= (L_{g_2 g_1})_*(X) = \mathrm{Ad}(g_2 g_1)(X) = (g_2 g_1) \cdot X. \end{aligned}$$

This completes the proof of the proposition. □

As claimed, $\cdot$ is an action of G on $\mathfrak{g}$. If H is a closed subgroup of G, then $\cdot$ is also an action of H on $\mathfrak{g}$. The action of H on $\mathfrak{g}$ can now be used to define an action of H on $\mathfrak{g}/\mathfrak{h}$. Begin by defining for $h \in H$ the map $\mathrm{Ad}(h) : \mathfrak{g}/\mathfrak{h} \to \mathfrak{g}/\mathfrak{h}$ by

$$\mathrm{Ad}(h)(X + \mathfrak{h}) = \mathrm{Ad}(h)(X) + \mathfrak{h}$$

for all $X + \mathfrak{h} \in \mathfrak{g}/\mathfrak{h}$.

Proposition 7.2. *If G is a Lie group with closed subgroup H, then for all $h \in H$ and $X + \mathfrak{h} \in \mathfrak{g}/\mathfrak{h}$,*

$$h \cdot (X + \mathfrak{h}) = \mathrm{Ad}(h)(X + \mathfrak{h})$$

defines an action of H on $\mathfrak{g}/\mathfrak{h}$.

Proof. Before this can be an action, it must be well-defined. If $X + \mathfrak{h} = Y + \mathfrak{h} \in \mathfrak{g}/\mathfrak{h}$, then $X - Y \in \mathfrak{h}$. If $h \in H$, then $\mathrm{Ad}(h) : \mathfrak{h} \to \mathfrak{h}$, so

$$\mathrm{Ad}(h)(X - Y) = \mathrm{Ad}(h)(X) - \mathrm{Ad}(h)(Y) \in \mathfrak{h}.$$

It follows that $\mathrm{Ad}(h)(X) + \mathfrak{h} = \mathrm{Ad}(h)(Y) + \mathfrak{h}$; hence $\mathrm{Ad}(h)(X + \mathfrak{h}) = \mathrm{Ad}(h)(Y + \mathfrak{h})$. Thus the map is well-defined.

If this is to be a group action, it must satisfy the conditions of Definition 3.1. First, if $e \in H$ is the identity element, then for all $X + \mathfrak{h} \in \mathfrak{g}/\mathfrak{h}$,

$$e \cdot (X + \mathfrak{h}) = \mathrm{Ad}(e)(X + \mathfrak{h}) = \mathrm{Ad}(e)(X) + \mathfrak{h} = X + \mathfrak{h},$$

as required. Secondly, let $h_1, h_2 \in H$ and $X + \mathfrak{h} \in \mathfrak{g}/\mathfrak{h}$. Then using previously found properties of Ad,

$$\begin{aligned} h_2 \cdot \big(h_1 \cdot (X + \mathfrak{h})\big) &= h_2 \cdot \big(\mathrm{Ad}(h_1)(X + \mathfrak{h})\big) = \mathrm{Ad}(h_2)\big(\mathrm{Ad}(h_1)(X) + \mathfrak{h}\big) \\ &= \mathrm{Ad}(h_2)\big(\mathrm{Ad}(h_1)(X)\big) + \mathfrak{h} \\ &= \mathrm{Ad}(h_2 h_1)(X) + \mathfrak{h} = \mathrm{Ad}(h_2 h_1)(X + \mathfrak{h}) \\ &= (h_2 h_1) \cdot (X + \mathfrak{h}). \end{aligned}$$

Therefore $\cdot$ is an action of H on $\mathfrak{g}/\mathfrak{h}$. $\square$

At this point, two actions of the Lie subgroup $H = G_{x_0}$ have been defined: (1) H acting on $T_{x_0}M$ and (2) H acting on $\mathfrak{g}/\mathfrak{h}$. For $h \in H$, these actions are defined as follows:

(1) if $v \in T_{x_0}M$, then
$$h \cdot v = \rho(h)(v) = (\mu_h)_*(v),$$

(2) if $X + \mathfrak{h} \in \mathfrak{g}/\mathfrak{h}$, then
$$h \cdot (X + \mathfrak{h}) = \mathrm{Ad}(h)(X + \mathfrak{h}) = \mathrm{Ad}(h)(X) + \mathfrak{h} = (L_h)_*(X) + \mathfrak{h}.$$

In order to show these actions are equivalent, begin by defining a map $\phi : \mathfrak{g}/\mathfrak{h} \to T_{x_0}M$ by
$$\phi(X + \mathfrak{h}) = (\mu_{x_0})_*(X_e) \tag{11}$$
for all $X + \mathfrak{h} \in \mathfrak{g}/\mathfrak{h}$. Recall from Section 3 the map $\mu_x : G \to M$ defined $\mu_x(g) = \mu(g, x)$ is smooth for any $x \in M$ and $g \in G$.

The first task is to show that ϕ is well-defined. Let $X + \mathfrak{h} = Y + \mathfrak{h} \in \mathfrak{g}/\mathfrak{h}$, then it follows that $X - Y \in \mathfrak{h}$; hence $(X - Y)_e = X_e - Y_e \in T_eH$. From Lemma 4.3, if $H = G_{x_0}$, then $T_eH = \ker(\mu_{x_0 *})$, so
$$(\mu_{x_0})_*(X_e - Y_e) = 0.$$
The map $(\mu_{x_0})_*$ is a linear map, so
$$(\mu_{x_0})_*(X_e) = (\mu_{x_0})_*(Y_e);$$
hence
$$\phi(X + \mathfrak{h}) = \phi(Y + \mathfrak{h}),$$
as required. Therefore the map ϕ is well-defined.

Theorem 7.3. *Let G be a Lie group acting on a manifold M by a transitive action μ. For any $x_0 \in M$, if $H = G_{x_0}$ is the isotropy subgroup of x_0 in G, then the map $\phi : \mathfrak{g}/\mathfrak{h} \to T_{x_0}M$ defined by*
$$\phi(X + \mathfrak{h}) = (\mu_{x_0})_*(X_e) \tag{11}$$
is an H-equivariant vector space isomorphism, that is
$$\phi\big(h \cdot (X + \mathfrak{h})\big) = h \cdot \phi(X + \mathfrak{h}) \tag{12}$$
for all $h \in H$ and $X + \mathfrak{h} \in \mathfrak{g}/\mathfrak{h}$.

Proof. The first step is to show that ϕ is an isomorphism. Begin by showing that it is a bijection. In order to show that it is one-to-one, consider $X + \mathfrak{h}, Y + \mathfrak{h} \in \mathfrak{g}/\mathfrak{h}$. Assume $\phi(X + \mathfrak{h}) = \phi(Y + \mathfrak{h})$, then
$$(\mu_{x_0})_*(X_e) = (\mu_{x_0})_*(Y_e),$$
so
$$0 = (\mu_{x_0})_*(X_e) - (\mu_{x_0})_*(Y_e) = (\mu_{x_0})_*(X_e - Y_e).$$

It follows that $X_e - Y_e = (X - Y)_e \in \ker(\mu_{x_0 *}) = T_e H$. Therefore $X - Y \in \mathfrak{h}$ and so $X + \mathfrak{h} = Y + \mathfrak{h}$; hence ϕ is one-to-one.

The proof that ϕ is onto $T_{x_0}M$ is just as straightforward. Let $v \in T_{x_0}M$. The image of $(\mu_{x_0})_*$ is $T_{x_0}M$ (once again using Lemma 4.3), so there exists a vector $X_e \in T_e G$ such that

$$v = (\mu_{x_0})_*(X_e).$$

If X is defined as the right invariant vector field on G such that X is X_e at $e \in G$, then

$$v = \phi(X + \mathfrak{h})$$

and so ϕ is onto $T_{x_0}M$, as required.

To be a vector space isomorphism, ϕ must be a vector space homomorphism, but this a direct result of $(\mu_{x_0})_*$ being linear:

$$\begin{aligned}
\phi\big((X + \mathfrak{h}) + (Y + \mathfrak{h})\big) &= \phi(X + Y + \mathfrak{h}) = (\mu_{x_0})_*\big((X + Y)_e\big) \\
&= (\mu_{x_0})_*(X_e + Y_e) = (\mu_{x_0})_*(X_e) + (\mu_{x_0})_*(Y_e) \\
&= \phi(X + \mathfrak{h}) + \phi(Y + \mathfrak{h}).
\end{aligned}$$

Therefore, ϕ is a vector space isomorphism and so the first part of the proof is complete.

It only remains to check Equation 12. First consider the left hand side of Equation 12:

$$\begin{aligned}
\phi\big(h \cdot (X + \mathfrak{h})\big) &= \phi\big(\mathrm{Ad}(h)(X + \mathfrak{h})\big) = \phi\big(\mathrm{Ad}(h)(X) + \mathfrak{h}\big) \\
&= (\mu_{x_0})_*\Big(\big(\mathrm{Ad}(h)(X)\big)_e\Big) = (\mu_{x_0})_*\Big(\big((L_h)_*(X)\big)_e\Big) \\
&= (\mu_{x_0})_*\big((L_h)_*(X_{h^{-1}})\big) = (\mu_{x_0} \circ L_h)_*(X_{h^{-1}}).
\end{aligned}$$

The vector field X is a right invariant vector field, so

$$X_{h^{-1}} = (R_{h^{-1}})_*(X_e);$$

hence

$$\begin{aligned}
\phi\big(h \cdot (X + \mathfrak{h})\big) &= (\mu_{x_0} \circ L_h)_*(X_{h^{-1}}) \\
&= (\mu_{x_0} \circ L_h)_*\big((R_{h^{-1}})_*(X_e)\big) \\
&= (\mu_{x_0} \circ L_h \circ R_{h^{-1}})_*(X_e).
\end{aligned}$$

Now consider the right hand side of Equation 12:

$$\begin{aligned}
h \cdot \phi(X + \mathfrak{h}) &= h \cdot (\mu_{x_0})_*(X_e) = \rho(h)(\mu_{x_0})_*(X_e) \\
&= (\mu_h)_*(\mu_{x_0})_*(X_e) = (\mu_h \circ \mu_{x_0})_*(X_e).
\end{aligned}$$

If Equation 12 is satisfied, then the left hand side,

$$(\mu_{x_0} \circ L_h \circ R_{h^{-1}})_*(X_e),$$

must equal the right hand side,

$$(\mu_h \circ \mu_{x_0})_*(X_e),$$

or

$$(\mu_{x_0} \circ L_h \circ R_{h^{-1}})_*(X_e) = (\mu_h \circ \mu_{x_0})_*(X_e). \tag{13}$$

So if Equation 13 holds then Equation 12 follows and the proof will be complete. Consider an arbitrary element $g \in G$. Then because $h \in G_{x_0}$ and consequently $h^{-1} \in G_{x_0}$, the left side of Equation 13 is

$$\begin{aligned}(\mu_{x_0} \circ L_h \circ R_{h^{-1}})(g) &= \mu_{x_0}(hgh^{-1}) = \mu(hgh^{-1}, x_0) \\ &= \mu\big(hg, \mu(h^{-1}, x_0)\big) = \mu(hg, x_0).\end{aligned}$$

Finally, the right side of Equation 13 is

$$\begin{aligned}(\mu_h \circ \mu_{x_0})(g) &= \mu_h\big(\mu_{x_0}(g)\big) = \mu\big(h, \mu(g, x_0)\big) \\ &= \mu(hg, x_0).\end{aligned}$$

Thus $\mu_{x_0} \circ L_h \circ R_{h^{-1}} = \mu_h \circ \mu_{x_0}$, so

$$(\mu_{x_0} \circ L_h \circ R_{h^{-1}})_* = (\mu_h \circ \mu_{x_0})_*,$$

as required. Therefore, Equation 13 holds, hence Equation 12 holds, and ϕ is H-equivariant. Thus all the necessary conditions have been met. □

This proof shows that H acts on $\mathfrak{g}/\mathfrak{h}$ as it acts on $T_{x_0}M$. This will be formalized in the following corollary.

Corollary 7.4. *Let G be a Lie group acting transitively on a manifold M by the smooth action μ. If $H = G_x$ for some $x \in M$ with linear isotropy representation ρ, then there is a one-to-one correspondence between ρH-invariant inner products on $T_{x_0}M$ and AdH-invariant inner products on $\mathfrak{g}/\mathfrak{h}$.*

8. The ad Action

The problem of finding invariant metrics on a manifold has so far been reduced to considering actions of a Lie group on a vector space. Before this can be further reduced to an action of a Lie algebra on a vector space, one more crucial tool must be introduced, the *exponential map*. The construction of the exponential map here follows that of [4], and requires the following theorem, given here without proof. (A full proof of this and the following theorems can be found in [4], pp. 101-104.)

Theorem 8.1. *Let G_1 and G_2 be Lie groups and let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be their respective Lie algebras. If G_1 is simply connected and $\psi : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a Lie algebra homomorphism, then their exists a unique group homomorphism $\phi : G_1 \to G_2$ such that $\psi = \phi_*$.*

Define $\mathfrak{L}(\mathbb{R})$ to be the Lie algebra of $\mathbb{R}$. The Lie algebra of $\mathbb{R}$ is one-dimensional, so call the one basis element $\frac{d}{dr}$; hence any element of $\mathfrak{L}(\mathbb{R})$ can be given as $\lambda\frac{d}{dr}$, where λ is any scalar. Let G be any Lie group with Lie algebra $\mathfrak{g}$. For any $X \in \mathfrak{g}$, the map $\psi : \mathfrak{L}(\mathbb{R}) \to \mathfrak{g}$ defined by

$$\psi\big(\lambda\frac{d}{dr}\big) = \lambda X$$

is a homomorphism of the Lie algebra of $\mathbb{R}$ into the Lie algebra $\mathfrak{g}$ of G. Theorem 8.1 then insures the existence of a unique homomorphism

$$\exp_X : \mathbb{R} \to G$$

such that $(\exp_X)_* = \psi$. As a note, the one-parameter subgroup given by $\exp_X : \mathbb{R} \to G$ is the unique integral curve of X at e.

Definition 8.1. *Let* $\exp_X : \mathbb{R} \to G$ *be as given above. Then the* exponential map

$$\exp : \mathfrak{g} \to G$$

is defined

$$\exp(X) = \exp_X(1)$$

for all $X \in \mathfrak{g}$.

The properties of the exp map are well known (see citewarner, for example) and will not be treated separately here; however, some properties will be needed in defining the adjoint action of a Lie algebra on a vector space which will be used throughout the rest of this paper. Those properties are listed now, without proof.

Theorem 8.2. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. The map* $\exp : \mathfrak{g} \to G$ *is smooth. Furthermore, if* $X \in \mathfrak{g}$*, then*

$$\exp(tX) = \exp_X(t)$$

for all $t \in \mathbb{R}$.

Theorem 8.3. *Let G_1 and G_2 be Lie groups and let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be their respective Lie algebras. If $\phi : G_1 \to G_2$ is a group homomorphism, then* $\phi \circ \exp = \exp \circ \phi_*$.

Theorem 8.4. *Let G be a Lie group with subgroup H. Let $\mathfrak{g}$ and $\mathfrak{h}$ be their respective Lie algebras. For $X \in \mathfrak{g}$,*

(1) if $X \in \mathfrak{h}$ then $\exp(tX) \in H$ *for all $t \in \mathbb{R}$, and*

(2) if $\exp(tX) \in H$ *for t in some open interval in $\mathbb{R}$, then $X \in \mathfrak{h}$.*

Given these properties of the exponential, recall $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ where $\mathrm{Ad}(g) : \mathfrak{g} \to \mathfrak{g}$ for $g \in G$ is defined $\mathrm{Ad}(g)(X) = (L_g)_* X$ or all $X \in \mathfrak{g}$. Now define

$$\mathrm{ad} : \mathfrak{g} \to \mathfrak{g}$$

as $\mathrm{ad} = \mathrm{Ad}_*$. Then for $X \in \mathfrak{g}$, $\mathrm{ad}(X) : \mathfrak{g} \to \mathfrak{g}$ is defined by

$$\mathrm{ad}(X)(Y) = \frac{d}{dt}\Big(\mathrm{Ad}\big(\exp_x(t)\big)(Y)\Big)\Big|_{t=0}$$

for all $Y \in \mathfrak{g}$, recalling that $\exp_X(t)$ is the unique integral curve of X at e.

From Theorem 8.3, the following equation must be true:

$$\mathrm{Ad} \circ \exp = \exp \circ \mathrm{ad}. \tag{14}$$

As a consequence,

$$\mathrm{Ad}\big(\exp(tX)\big) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \mathrm{ad}(X)^k. \tag{15}$$

These equations will prove very useful.

Proposition 8.5. *Let G be a lie group with Lie algebra $\mathfrak{g}$. If $X \in \mathfrak{g}$, then*

$$\mathrm{ad}(X)(Y) = [X, Y] \tag{16}$$

for all $Y \in \mathfrak{g}$, where $[,]$ is the Lie algebra bracket on $\mathfrak{g}$.

Proof. Begin by fixing $X \in \mathfrak{g}$. For simplicity, let $\gamma(t) := \exp_X(t)$, the unique integral curve of X at $e \in G$. Let $Y \in \mathfrak{g}$. Both $\mathrm{ad}(X)(Y)$ and $[X, Y]$ are elements of $\mathfrak{g}$, hence right invariant vector fields, therefore it is enough to show equality at the identity. Following from definition,

$$\mathrm{ad}(X)(Y)_e = \frac{d}{dt}\Big(\mathrm{Ad}\big(\gamma(t)\big)(Y)\Big)_e\Big|_{t=0} = \frac{d}{dt}\Big(\big(L_{\gamma(t)}\big)_*(Y)\Big)_e\Big|_{t=0}. \tag{17}$$

Note, however, that

$$\Big(\big(L_{\gamma(t)}\big)_*(Y)\Big)_e = \big(L_{\gamma(t)}\big)_*\big(Y_{\gamma(-t)}\big) = \big(L_{\gamma(t)}\big)_*\big(Y_{L_{\gamma(-t)}}\big)$$

and

$$\frac{d}{dt}\big(L_{\gamma(t)}\big)_*\big(Y_{L_{\gamma(-t)}}\big)\Big|_{t=0} = \big(\mathcal{L}_X Y\big)_e.$$

This last expression is the Lie derivative of Y with respect to X, which in the case of vector fields is merely the bracket operation. Therefore, substituting into Equation 17

$$\mathrm{ad}(X)(Y)_e = \big(\mathcal{L}_X Y\big)_e = [X, Y]_e,$$

as required. $\square$

Now it will be shown that, given a Lie group G with a connected subgroup H, there exists a one-to-one correspondence between inner products on $\mathfrak{g}/\mathfrak{h}$ invariant under action by H and inner products on $\mathfrak{g}/\mathfrak{h}$ invariant under action by $\mathfrak{h}$, the Lie algebra of H.

Definition 8.2. *Let $\mathfrak{g}$ be a (real) Lie algebra and V a vector space. A mapping $\mu : \mathfrak{g} \times V \to V$ is said to be a* Lie algebra action *of $\mathfrak{g}$ on the vector space V if the following two conditions are satisfied:*

(1) If $X_1, X_2 \in \mathfrak{g}$ and $u \in V$, then for $\alpha \in \mathbb{R}$

$$\mu(\alpha X_1 + X_2, u) = \alpha\mu(X_1, u) + \mu(X_2, u).$$

(2) If $X_1, X_2 \in \mathfrak{g}$, then for all $u \in V$

$$\mu(X_1, \mu(X_2, u)) - \mu(X_2, \mu(X_1, u)) = \mu([X_1, X_2], u).$$

The Lie algebra $\mathfrak{g}$ is said to act *on the vector space V by the action μ. For simplicity, $\mu(X, u)$ is often written $X \cdot u$.*

Note that the Lie bracket operation $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ defines an action of a Lie algebra on itself. Property (1) is clear from the definition of the bracket, and Property (2) follows directly from the Jacobi identity.

Proposition 8.6. *If $\mathfrak{g}$ is a Lie algebra, then for all $X, Y \in \mathfrak{g}$*

$$X \cdot Y = \mathrm{ad}(X)(Y)$$

defines an action of $\mathfrak{g}$ on $\mathfrak{g}$.

Proof. The proof follows directly from Proposition 8.5. $\square$

Thus $\cdot$ provides an action of $\mathfrak{g}$ on $\mathfrak{g}$, hence an action of $\mathfrak{h}$ on $\mathfrak{g}$. In turn, this can now be used to define an action of $\mathfrak{h}$ on $\mathfrak{g}/\mathfrak{h}$. Begin by defining for $X \in \mathfrak{g}$ the map $\mathrm{ad}(X) : \mathfrak{g}/\mathfrak{h} \to \mathfrak{g}/\mathfrak{h}$ by

$$\mathrm{ad}(X)(Y + \mathfrak{h}) = \mathrm{ad}(X)(Y) + \mathfrak{h}$$

for all $Y + \mathfrak{h} \in \mathfrak{g}/\mathfrak{h}$.

Proposition 8.7. *If $\mathfrak{g}$ is a Lie algebra with Lie subalgebra $\mathfrak{h}$, then for all $X \in \mathfrak{h}$ and $Y + \mathfrak{h} \in \mathfrak{g}/\mathfrak{h}$*

$$X \cdot (Y + \mathfrak{h}) = \mathrm{ad}(X)(Y + \mathfrak{h})$$

defines an action of $\mathfrak{h}$ on $\mathfrak{g}/\mathfrak{h}$.

Proof. Before this can be an action, it must be well-defined. If $Y + \mathfrak{h} = Z + \mathfrak{h} \in \mathfrak{g}/\mathfrak{h}$, then $Y - Z \in \mathfrak{h}$. By assumption, $X \in \mathfrak{h}$ as well, so $[X, Y - Z] \in \mathfrak{h}$ because $\mathfrak{h}$ is subalgebra. By bilinearity of the bracket operation,

$$[X, Y - Z] = [X, Y] - [X - Z] \in \mathfrak{h}.$$

It follows that $[X, Y] + \mathfrak{h} = [X, Z] + \mathfrak{h}$, hence $\mathrm{ad}(X)(Y) + \mathfrak{h} = \mathrm{ad}(X)(Z) + \mathfrak{h}$ and so

$$\mathrm{ad}(X)(Y + \mathfrak{h}) = \mathrm{ad}(X)(Z + \mathfrak{h}),$$

as required. Thus the map is well-defined.

Now it is necessary to check the two conditions of Definition 8.2. First, assume $X_1, X_2 \in \mathfrak{h}$ and $\alpha \in \mathbb{R}$. For all $Y + \mathfrak{h} \in \mathfrak{g}/\mathfrak{h}$,

$$\begin{aligned}
(\alpha X_1 + X_2) \cdot (Y + \mathfrak{h}) &= \mathrm{ad}(\alpha X_1 + X_2)(Y + \mathfrak{h}) = \mathrm{ad}(\alpha X_1 + X_2)(Y) + \mathfrak{h} \\
&= \alpha \mathrm{ad}(X_1)(Y) + \mathrm{ad}(X_2)(Y) + \mathfrak{h} = \alpha \mathrm{ad}(X_1)(Y + \mathfrak{h}) + \mathrm{ad}(X_2)(Y + \mathfrak{h}) \\
&= \alpha\big(X_1 \cdot (Y + \mathfrak{h})\big) + \big(X_2 \cdot (Y + \mathfrak{h})\big),
\end{aligned}$$

as required.

Secondly, assume $X_1, X_2 \in \mathfrak{h}$. For all $Y + \mathfrak{h} \in \mathfrak{g}/\mathfrak{h}$,

$$\begin{aligned}
&X_1 \cdot \big(X_2 \cdot (Y + \mathfrak{h})\big) - X_2 \cdot \big(X_1 \cdot (Y + \mathfrak{h})\big) \\
&\quad = \mathrm{ad}(X_1)\big(\mathrm{ad}(X_2)(Y + \mathfrak{h})\big) - \mathrm{ad}(X_2)\big(\mathrm{ad}(X_1)(Y + \mathfrak{h})\big) \\
&\quad = \mathrm{ad}(X_1)\big(\mathrm{ad}(X_2)(Y) + \mathfrak{h}\big) - \mathrm{ad}(X_2)\big(\mathrm{ad}(X_1)(Y) + \mathfrak{h}\big) \\
&\quad = \mathrm{ad}(X_1)\big(\mathrm{ad}(X_2)(Y)\big) + \mathfrak{h} - \mathrm{ad}(X_2)\big(\mathrm{ad}(X_1)(Y)\big) + \mathfrak{h}.
\end{aligned}$$

It has already been shown, however, that ad is an action of $\mathfrak{g}$ on itself, so

$$\mathrm{ad}(X_1)\big(\mathrm{ad}(X_2)(Y)\big) - \mathrm{ad}(X_2)\big(\mathrm{ad}(X_1)(Y)\big) = \mathrm{ad}\big([X_1, X_2]\big)(Y).$$

Therefore,

$$\begin{aligned} & X_1 \cdot \big(X_2 \cdot (Y + \mathfrak{h})\big) - X_2 \cdot \big(X_1 \cdot (Y + \mathfrak{h})\big) \\ = \;& \operatorname{ad}(X_1)\big(\operatorname{ad}(X_2)(Y)\big) - \operatorname{ad}(X_2)\big(\operatorname{ad}(X_1)(Y)\big) + \mathfrak{h} \\ = \;& \operatorname{ad}\big([X_1, X_2]\big)(Y) + \mathfrak{h} = \operatorname{ad}\big([X_1, X_2]\big)(Y + \mathfrak{h}) \\ = \;& \big([X_1, X_2]\big) \cdot (Y + \mathfrak{h}). \end{aligned}$$

Thus, $\cdot$ does define an action of $\mathfrak{h}$ on $\mathfrak{g}/\mathfrak{h}$. □

Definition 8.3. *Let $\mathfrak{g}$ be a Lie algebra. A quadratic form Q on a vector space V is said to be* $\mathfrak{g}$-invariant *if there is an action $\cdot : \mathfrak{g} \times V \to V$ such that*

$$Q(X \cdot u, v) + Q(u, X \cdot v) = 0$$

for all $u, v \in V$ and $X \in \mathfrak{g}$. If this equation holds for $X \in \mathfrak{g}$, then it is written $X \cdot Q = Q$.

The same is said for G-invariance, where G is a Lie group acting on a vector space, with $g \in G$ replacing $X \in \mathfrak{g}$ in the definition above.

Let the Lie algebra $\mathfrak{g}$ act on the vector space V. Let $\{u_1, \ldots, u_q\}$ be a basis for the vector space V and $X \cdot u_i = \alpha_i^j u_j$, for $X \in \mathfrak{g}$. If Q is a $\mathfrak{g}$-invariant quadratic form on V, then

$$\alpha_i^k Q_{kj} + \alpha_j^k Q_{ik} = 0 \iff A^T Q + QA = 0, \tag{18}$$

where A is the matrix having components $A_i^j = \alpha_i^j$.

Recall that an inner product is a non-degenerate quadratic form. Thus for a Lie algebra $\mathfrak{g}$ with subalgebra $\mathfrak{h}$ an inner product η on the vector space $\mathfrak{g}/\mathfrak{h}$ is said to be ad $\mathfrak{h}$-invariant (i.e. invariant under the ad action) if

$$\eta\Big(\operatorname{ad}(X)(Y + \mathfrak{h}), Z + \mathfrak{h}\Big) + \eta\Big(Y + \mathfrak{h}, \operatorname{ad}(X)(Z + \mathfrak{h})\Big) = 0 \tag{19}$$

for all $X \in \mathfrak{h}$ and $Y + \mathfrak{h}, Z + \mathfrak{h} \in \mathfrak{g}/\mathfrak{h}$.

Theorem 8.8. *Let G be a Lie group with subgroup H where G and H have Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively. Let η be an inner product on $\mathfrak{g}/\mathfrak{h}$. If η is $\operatorname{Ad}H$-invariant, then it is $\operatorname{ad}\mathfrak{h}$-invariant.*

Proof. Assume η on $\mathfrak{g}/\mathfrak{h}$ is $\operatorname{Ad}H$-invariant. Then, by Definition 8.3,

$$\eta\Big(\operatorname{Ad}(h)(Y + \mathfrak{h}), Z + \mathfrak{h}\Big) + \eta\Big(Y + \mathfrak{h}, \operatorname{Ad}(h)(Z + \mathfrak{h})\Big) = 0$$

for all $h \in H$ and $Y + \mathfrak{h}, Z + \mathfrak{h} \in \mathfrak{g}/\mathfrak{h}$. Let $X \in \mathfrak{h}$, then

$$\operatorname{ad}(X) = \frac{d}{dt} \operatorname{Ad}\Big(\exp_X(t)\Big)\bigg|_{t=0}$$

for $t \in \mathrm{Dom}(\exp_X)$. Thus for $Y+\mathfrak{h}, Z+\mathfrak{h} \in \mathfrak{g}/\mathfrak{h}$

$$\begin{aligned}
&\eta\Big(\mathrm{ad}(X)(Y+\mathfrak{h}), Z+\mathfrak{h}\Big) + \eta\Big(Y+\mathfrak{h}, \mathrm{ad}(X)(Z+\mathfrak{h})\Big) \\
&= \eta\Big(\frac{d}{dt}\mathrm{Ad}\big(\exp_X(t)\big)(Y+\mathfrak{h})\Big|_{t=0}, Z+\mathfrak{h}\Big) + \eta\Big(Y+\mathfrak{h}, \frac{d}{dt}\mathrm{Ad}\big(\exp_X(t)\big)(Z+\mathfrak{h})\Big|_{t=0}\Big) \\
&= \frac{d}{dt}\eta\Big(\mathrm{Ad}\big(\exp_X(t)\big)(Y+\mathfrak{h}), Z+\mathfrak{h}\Big)\Big|_{t=0} + \frac{d}{dt}\eta\Big(Y+\mathfrak{h}, \mathrm{Ad}\big(\exp_X(t)\big)(Z+\mathfrak{h})\Big)\Big|_{t=0} \\
&= \frac{d}{dt}\Big[\eta\Big(\mathrm{Ad}\big(\exp_X(t)\big)(Y+\mathfrak{h}), Z+\mathfrak{h}\Big) + \eta\Big(Y+\mathfrak{h}, \mathrm{Ad}\big(\exp_X(t)\big)(Z+\mathfrak{h})\Big)\Big]_{t=0}.
\end{aligned}$$

Note, however, that $\exp_X(t) = \exp(tX) \in H$ for all t by Theorem 8.4. Thus,

$$\eta\Big(\mathrm{ad}(X)(Y+\mathfrak{h}), Z+\mathfrak{h}\Big) + \eta\Big(Y+\mathfrak{h}, \mathrm{ad}(X)(Z+\mathfrak{h})\Big) = \frac{d}{dt}(0)\Big|_{t=0} = 0.$$

Therefore η is $\mathrm{ad}\mathfrak{h}$-invariant. □

Lemma 8.9. *Let G be a Lie group with subgroup H where G and H have Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively. Let η be an inner product on $\mathfrak{g}/\mathfrak{h}$. If η is $\mathrm{ad}\mathfrak{h}$-invariant, then there exists a neighborhood U of H such that η is $\mathrm{Ad}H$-invariant on U; i.e. $g \cdot \eta = \eta$ for all $g \in U$.*

Proof. Assume η is $\mathrm{ad}\mathfrak{h}$-invariant. Then

$$\eta\Big(\mathrm{ad}(X)(Y+\mathfrak{h}), Z+\mathfrak{h}\Big) + \eta\Big(Y+\mathfrak{h}, \mathrm{ad}(X)(Z+\mathfrak{h})\Big) = 0$$

for all $X \in \mathfrak{h}$ and $Y+\mathfrak{h}, Z+\mathfrak{h} \in \mathfrak{g}/\mathfrak{h}$. Let $U = \exp(\mathfrak{h})$. For all $g \in U$, there exists an $X \in \mathfrak{h}$ such that $g = \exp(X)$. Using this and applying Equation 15 (with $t=1$),

$$\begin{aligned}
&\eta\Big(\mathrm{Ad}(g)(Y+\mathfrak{h}), Z+\mathfrak{h}\Big) + \eta\Big(Y+\mathfrak{h}, \mathrm{Ad}(g)(Z+\mathfrak{h})\Big) \\
&= \eta\Big(\mathrm{Ad}\big(\exp(X)\big)(Y+\mathfrak{h}), Z+\mathfrak{h}\Big) + \eta\Big(Y+\mathfrak{h}, \mathrm{Ad}\big(\exp(X)\big)(Z+\mathfrak{h})\Big) \\
&= \eta\Big(\sum_{k=0}^{\infty}\frac{1}{k!}\mathrm{ad}(X)^k(Y+\mathfrak{h}), Z+\mathfrak{h}\Big) + \eta\Big(Y+\mathfrak{h}, \sum_{k=0}^{\infty}\frac{t^k}{k!}\mathrm{ad}(X)^k(Z+\mathfrak{h})\Big) \\
&= \sum_{k=0}^{\infty}\frac{1}{k!}\eta\Big(\mathrm{ad}(X)^k(Y+\mathfrak{h}), Z+\mathfrak{h}\Big) + \sum_{k=0}^{\infty}\frac{1}{k!}\eta\Big(Y+\mathfrak{h}, \mathrm{ad}(X)^k(Z+\mathfrak{h})\Big) \\
&= \sum_{k=0}^{\infty}\frac{1}{k!}\Big[\eta\Big(\mathrm{ad}(X)^k(Y+\mathfrak{h}), Z+\mathfrak{h}\Big) + \eta\Big(Y+\mathfrak{h}, \mathrm{ad}(X)^k(Z+\mathfrak{h})\Big)\Big]
\end{aligned}$$

Note, however, that $\mathrm{ad}(X)^k(Y+\mathfrak{h}) = \mathrm{ad}(X)\Big(\mathrm{ad}(X)^{k-1}(Y+\mathfrak{h})\Big)$, where $\mathrm{ad}(X)^{k-1}(Y+\mathfrak{h}) \in \mathfrak{g}/\mathfrak{h}$. Thus,

$$\eta\Big(\mathrm{Ad}(g)(Y+\mathfrak{h}), Z+\mathfrak{h}\Big) + \eta\Big(Y+\mathfrak{h}, \mathrm{Ad}(g)(Z+\mathfrak{h})\Big) = \sum_{k=0}^{\infty}\frac{1}{k!}(0) = 0.$$

Therefore η is $\mathrm{Ad}H$-invariant on $U = \exp(\mathfrak{h})$. □

Lemma 8.10. *Let G be a Lie group and U any open neighborhood of the identity in G. If G is connected, then for any $g \in G$ there exist $g_1, \ldots, g_k \in U$ such that*

$$g = g_1 \cdots g_k.$$

Proof. For a proof see [4], pp. 93-94. □

Lemma 8.11. *Let G be Lie group which acts on a vector space V and let η be an inner product on V. If there exits a neighborhood of the identity U in G such that η is U-invariant under the action of G on V, then if G is connected, η is G-invariant.*

Proof. Let $\cdot$ be the action of G on V and assume the conditions of the lemma hold. G is connected, so by Lemma 8.10 for any $g \in G$ there exist $g_1, \ldots, g_k \in U$ such that

$$g = g_1 \cdots g_k.$$

By assumption, η on V is U-invariant, so

$$g_i \cdot \eta = \eta$$

for all $g_i \in U$. Therefore

$$\begin{aligned} g \cdot \eta &= (g_1 \cdots g_k) \cdot \eta = (g_1 \cdots g_{k-1}) \cdot (g_k \cdot \eta) = (g_1 \cdots g_{k-1}) \cdot \eta \\ &= \ldots = (g_1 g_2) \cdot \eta = g_1 \cdot (g_2 \cdot \eta) = g_1 \cdot \eta \\ &= \eta. \end{aligned}$$

Thus $g \cdot \eta = \eta$ for any $g \in G$, as required, hence η is G-invariant. □

Theorem 8.12. *Let G be a Lie group with subgroup H where G and H have Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively. Let η be an inner product on $\mathfrak{g}/\mathfrak{h}$. If η is $\mathrm{ad}\mathfrak{h}$-invariant and H is connected, then η is $\mathrm{Ad}H$-invariant.*

Proof. The result follows directly from Lemmas 8.9 and 8.11. □

Corollary 8.13. *Let G be a Lie group with connected subgroup H where G and H have Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively. An inner product η on $\mathfrak{g}/\mathfrak{h}$ is $\mathrm{Ad}H$-invariant if and only if it is $\mathrm{ad}\mathfrak{h}$-invariant.*

Proof. This is a direct consequence of Theorems 8.8 and 8.12. □

As a consequence of Corollary 8.13, the problem of finding invariant metrics on manifolds is reduced to a purely algebraic problem.

9. $\mathfrak{g}$-Equivariant Isomorphisms

From this point on, the theory presented will be purely algebraic; however, it shall be implicit that the Lie algebras in question arose from the set of right-invariant vector fields on a Lie group, hence the labels $\mathfrak{g}$ and $\mathfrak{h}$ will be used.

Definition 9.1. *A* Lie algebra pair *($\mathfrak{g}$, $\mathfrak{h}$) is a Lie algebra $\mathfrak{g}$ with a Lie subalgebra $\mathfrak{h}$.*

Two Lie algebra pairs $(\mathfrak{g}_1, \mathfrak{h}_1)$ and $(\mathfrak{g}_2, \mathfrak{h}_2)$ are said to be isomorphic, or equivalent, if there is an isomorphism of Lie algebras, $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ such that $\phi(\mathfrak{h}_1) = \mathfrak{h}_2$. A special case of the above is $\mathfrak{g}_1 = \mathfrak{g}_2 = \mathfrak{g}$, which gives the following: $(\mathfrak{g}, \mathfrak{h}_1) \sim (\mathfrak{g}, \mathfrak{h}_2)$ if there exists a Lie algebra *automorphism* $\phi : \mathfrak{g} \to \mathfrak{g}$ such that $\phi(\mathfrak{h}_1) = \mathfrak{h}_2$.

Definition 9.2. *Given a vector X in a Lie algebra $\mathfrak{g}$, define* $\mathrm{Ad}(X) : \mathfrak{g} \to \mathfrak{g}$ *by* $\mathrm{Ad}(X) = \mathrm{Ad}\big(\exp(X)\big)$*. It follows that* $\mathrm{Ad}(X) = \exp\big(\mathrm{ad}(X)\big)$*.* $\mathrm{Ad}(X)$ *is an automorphism of the Lie algebra* $\mathfrak{g}$*. In fact,* $\mathrm{Ad}(\mathfrak{g}) \subseteq \mathrm{Aut}(\mathfrak{g})$ *as a normal subgroup, called the* inner automorphisms*. The quotient* $\mathrm{Aut}(\mathfrak{g})/\mathrm{Ad}(\mathfrak{g})$ *is called the* outer automorphisms*, which is labeled* $\mathrm{Out}(\mathfrak{g})$*.*

Theorem 9.1. *Let $(\mathfrak{g}, \mathfrak{h}_1)$ and $(\mathfrak{g}, \mathfrak{h}_2)$ be two Lie algebra pairs. If $\phi : \mathfrak{g} \to \mathfrak{g}$ is a Lie algebra automorphism such that $\phi(\mathfrak{h}_1) = \mathfrak{h}_2$, then ϕ induces a $\mathfrak{g}$-equivariant isomorphism $\tilde{\phi} : \mathfrak{g}/\mathfrak{h}_1 \to \mathfrak{g}/\mathfrak{h}_2$.*

Proof. Define the induced mapping $\tilde{\phi} : \mathfrak{g}/\mathfrak{h}_1 \to \mathfrak{g}/\mathfrak{h}_2$ by

$$\tilde{\phi}(Y + \mathfrak{h}_1) = \phi(Y) + \mathfrak{h}_2$$

for all $Y + \mathfrak{h}_1 \in \mathfrak{g}/\mathfrak{h}_1$. In order to show this mapping is well-defined, it is necessary to show $\tilde{\phi}(Y + \mathfrak{h}_1) = \tilde{\phi}(Y' + \mathfrak{h}_1)$ for any two $Y, Y' \in \mathfrak{g}$ in the same coset. Assume $Y + \mathfrak{h}_1 = Y' + \mathfrak{h}_1$. Then $Y - Y' \in \mathfrak{h}_1$. By assumption $\phi(\mathfrak{h}_1) = \mathfrak{h}_2$, so it follows that $\phi(Y - Y') \in \mathfrak{h}_2$. It was also assumed that ϕ was a Lie algebra automorphism, hence a homomorphism, so $\phi(Y - Y') = \phi(Y) - \phi(Y')$. Thus $\phi(Y)\mathfrak{h}_2 = \phi(Y')\mathfrak{h}_2$, implying $\tilde{\phi}(Y + \mathfrak{h}_1) = \tilde{\phi}(Y' + \mathfrak{h}_1)$, as required.

To show $\tilde{\phi}$ is $\mathfrak{g}$-equivariant, consider the ad action of $\mathfrak{g}$ on both $\mathfrak{g}/\mathfrak{h}_1$ and $\mathfrak{g}/\mathfrak{h}_2$. If $Y + \mathfrak{h} \in \mathfrak{g}/\mathfrak{h}_1$, then for any $X \in \mathfrak{g}$

$$\tilde{\phi}\big(X \cdot (Y + \mathfrak{h}_1)\big) = \tilde{\phi}\big([X, Y] + \mathfrak{h}_1\big) = \phi\big([X, Y]\big) + \mathfrak{h}_2.$$

The map ϕ is a Lie algebra homomorphism, so

$$\phi\big([X, Y]\big) + \mathfrak{h}_2 = [\phi(X), \phi(Y)] + \mathfrak{h}_2 = \phi(X) \cdot \big(\phi(Y) + \mathfrak{h}_2\big) = \phi(X) \cdot \tilde{\phi}\big(Y + \mathfrak{h}_1\big).$$

Therefore $\tilde{\phi}\big(X \cdot (Y + \mathfrak{h}_1)\big) = \phi(X) \cdot \tilde{\phi}\big(Y + \mathfrak{h}_1\big)$ and so $\tilde{\phi}$ is $\mathfrak{g}$-equivariant.

It only remains to show that $\tilde{\phi}$ is bijective. Assume $\tilde{\phi}(Y + \mathfrak{h}_1) = \tilde{\phi}(Y' + \mathfrak{h}_1)$ for some $Y + \mathfrak{h}_1, Y' + \mathfrak{h}_1 \in \mathfrak{g}/\mathfrak{h}_2$. It follows that $\phi(Y) + \mathfrak{h}_2 = \phi(Y') + \mathfrak{h}_2$, so

$$\phi(Y) - \phi(Y') = \phi(Y - Y') \in \mathfrak{h}_2,$$

so $Y - Y' \in \mathfrak{h}_1$, hence $Y + \mathfrak{h}_1 = Y' + \mathfrak{h}_1$. Thus $\tilde{\phi}$ is one-to-one.

Now consider $\tilde{Y} + \mathfrak{h}_2 \in \mathfrak{g}/\mathfrak{h}_2$. The map $\phi|_{\mathfrak{h}_1} : \mathfrak{h}_1 \to \mathfrak{h}_2$ is an isomorphism, thus there exists a $Y \in \mathfrak{h}_1$ such that $\tilde{Y} = \phi(Y)$. Consequently,

$$\tilde{\phi}(Y + \mathfrak{h}_1) = \phi(Y) + \mathfrak{h}_2 = \tilde{Y} + \mathfrak{h}_2.$$

Thus $\tilde{\phi}$ is onto and so a bijection. Therefore $\tilde{\phi} : \mathfrak{g}/\mathfrak{h}_1 \to \mathfrak{g}/\mathfrak{h}_2$ is a $\mathfrak{g}$-equivariant isomorphism. □

Definition 9.3. *The Lie algebra pair $(\mathfrak{g}, \mathfrak{h})$ admits a* reductive complement *if there is a subspace $\mathfrak{m}$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ (vector space direct sum), where*

$$[\mathfrak{m}, \mathfrak{h}] \subseteq \mathfrak{m}.$$

In this case, the Lie algebra pair $(\mathfrak{g}, \mathfrak{h})$ is said to be a reductive pair*.*

Definition 9.4. *The Lie algebra pair* $(\mathfrak{g}, \mathfrak{h})$ *is said to be a* symmetric pair *if there exists a reductive complement* $\mathfrak{m}$ *of* $\mathfrak{h}$ *in* $\mathfrak{g}$ *such that*

$$[\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{h}.$$

In this case, $\mathfrak{m}$ *is sometimes called a* symmetric complement*.*

Definition 9.5. *For any Lie algebra* $\mathfrak{g}$*, a subset* $\mathfrak{m}$ *is said to be an* ideal *of* $\mathfrak{g}$ *if*

$$[\mathfrak{m}, \mathfrak{g}] \subseteq \mathfrak{m}.$$

In particular, if the Lie algebra pair $(\mathfrak{g}, \mathfrak{h})$ admits a compliment $\mathfrak{m}$ such that $\mathfrak{m}$ is an ideal of $\mathfrak{g}$, then $\mathfrak{m}$ is a reductive compliment of $\mathfrak{h}$ in $\mathfrak{g}$.

Proposition 9.2. *If the Lie algebra pair* $(\mathfrak{g}, \mathfrak{h})$ *admits a reductive complement* $\mathfrak{m}$*, then there exists a vector space isomorphism* $\psi : \mathfrak{g}/\mathfrak{h} \to \mathfrak{m}$ *which is invariant under the* ad$\mathfrak{h}$ *action.*

Proof. Assume $(\mathfrak{g}, \mathfrak{h})$ is a reductive Lie algebra pair. By definition there exists an $\mathfrak{m}$ such that $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ and $[\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m}$. The complement $\mathfrak{m} = \mathfrak{g} - \mathfrak{h}$ has the same dimension as $\mathfrak{g}/\mathfrak{h}$, so they must be isomorphic as vector spaces. Let $\{Y_1 + \mathfrak{h}, \ldots, Y_k + \mathfrak{h}\}$ be basis for $\mathfrak{g}/\mathfrak{h}$. For $i = 1, \ldots, k$

$$Y_i + \mathfrak{h} \neq \mathfrak{h} \implies Y_i \notin \mathfrak{h}.$$

Therefore $Y_i \in \mathfrak{m}$ for all i. If the set $\{Y_1, \ldots, Y_k\}$ did not form a basis for $\mathfrak{m}$, then the Y_i would be linearly dependant. If this were the case, then there would exist scalars $\alpha^i \in \mathbb{R}$ such that

$$\alpha^i Y_i = 0 \implies (\alpha^i Y_i) + \mathfrak{h} = \mathfrak{h} \implies \alpha^i (Y_i + \mathfrak{h}) = \mathfrak{h},$$

a contradiction of the fact that the $Y_i + \mathfrak{h}$ formed a basis for $\mathfrak{g}/\mathfrak{h}$. Thus $\{Y_1, \ldots, Y_k\}$ forms a basis for $\mathfrak{m}$.

Define $\psi : \mathfrak{g}/\mathfrak{h} \to \mathfrak{m}$ on the basis elements:

$$\psi(Y_i + \mathfrak{h}) = Y_i$$

for $i = 1, \ldots, k$. This is a map defined on the bases, hence it is a vector space isomorphism. It only remains to show that ψ preserves the action of $\mathfrak{h}$ by ad. Let $X \in \mathfrak{h}$ and $Y + \mathfrak{h} \in \mathfrak{g}/\mathfrak{h}$. Using the previously given basis for $\mathfrak{g}/\mathfrak{h}$,

$$Y + \mathfrak{h} = a^i (Y_i + \mathfrak{h}) = (a^i Y_i) + \mathfrak{h}$$

for some set of $a^i \in \mathbb{R}$. Thus

$$\begin{aligned} \mathrm{ad}(X)(Y + \mathfrak{h}) &= \mathrm{ad}(X)\big((a^i Y_i) + \mathfrak{h}\big) \\ &= \mathrm{ad}(X)(a^i Y_i) + \mathfrak{h} \\ &= [X, a^i Y_i] + \mathfrak{h} \\ &= a^i [X, Y_i] + \mathfrak{h} \end{aligned}$$

By assumption, however, $X \in \mathfrak{h}$ and $Y_i \in \mathfrak{m}$ implies that $[X, Y_i] \in \mathfrak{m}$. It follows that for each $i = 1, \ldots, k$

$$[X, Y_i] = b_i^j Y_j$$

for some set of $b_i^j \in \mathbb{R}$. It follows that

$$\begin{aligned}\psi\Big(\mathrm{ad}(X)(Y+\mathfrak{h})\Big) &= \psi\Big(a^i[X,Y_i]+\mathfrak{h}\Big) = \psi\Big(a^i(b_i^jY_j)+\mathfrak{h}\Big) \\ &= \psi\Big(a^ib_i^j(Y_j+\mathfrak{h})\Big) = a^ib_i^j\Big(\psi(Y_j+\mathfrak{h})\Big) \\ &= a^ib_i^j(Y_j).\end{aligned}$$

Using some straightforward properties of linear algebra,

$$a^ib_i^j(Y_j) = a^i(b_i^jY_j) = a^i[X,Y_i] = [X,a^iY_i] = [X,Y] = \mathrm{ad}(X)(Y).$$

Therefore

$$\psi\Big(\mathrm{ad}(X)(Y+\mathfrak{h})\Big) = \mathrm{ad}(X)(Y)$$

as required. $\square$

Loosely speaking, this result shows $\mathfrak{g}/\mathfrak{h}$ can be identified with $\mathfrak{m}$, if $\mathfrak{m}$ is a reductive complement of $\mathfrak{h}$ in $\mathfrak{g}$.

Definition 9.6. *Given an inner product, η, on a vector space V, the restriction to the subspace U, $\eta|_U$, is non-degenerate if $u_1 \in U$ and $\eta(u_1,u_2)=0$ for all $u_2 \in U$ implies that $u_1 = 0$.*

Proposition 9.3. *Let $\mathfrak{g}$ be a Lie algebra with subalgebra $\mathfrak{h}$. If η is an* ad$\mathfrak{h}$*-invariant inner product on $\mathfrak{g}$ and $\eta|_{\mathfrak{h}}$ is non-degenerate, then $(\mathfrak{g},\mathfrak{h})$ admits a reductive complement $\mathfrak{m}$ such that $\eta|_{\mathfrak{m}}$ is non-degenerate.*

Proof. Assume that the dimensions of $\mathfrak{g}$ and $\mathfrak{h}$ are n and k, respectively. If either $k=0$ or $k=n$, then the proposition is trivially true. It can therefore be assumed that $0<k<n$. Let $\mathfrak{m}$ be the complement of $\mathfrak{h}$ in $\mathfrak{g}$. By hypothesis, η is non-degenerate on $\mathfrak{g}$. It follows that a basis $\{X_1,\dots,X_n\}$ for $\mathfrak{g}$ can be constructed so that

$$\eta(X_i,X_j) = \pm\delta_{ij}$$

for $i,j \in \{1,\dots,n\}$. Similarly, $\eta|_{\mathfrak{h}}$ is non-degenerate, so a basis $\{Y_1,\dots,Y_k\}$ can be constructed for $\mathfrak{h}$ such that

$$\eta(Y_i,Y_j) = \pm\delta_{ij}$$

for $i,j \in \{1,\dots,k\}$. Consider $Y_1 \in \mathfrak{g}$. The set $\{X_1,\dots,X_n\}$ is basis for $\mathfrak{g}$, so

$$Y_1 = \alpha_1^1X_1 + \cdots + \alpha_1^nX_n$$

where $\alpha_1^1,\dots,\alpha_1^n \in \mathbb{R}$ and $\alpha_1^j \neq 0$ for at least one $j=1,\dots,n$. Assume without loss of generality that $\alpha_1^1 \neq 0$. Then

$$X_1 = \frac{1}{\alpha_1^1}Y_1 - \frac{\alpha_1^2}{\alpha_1^1}X_2 - \cdots - \frac{\alpha_1^n}{\alpha_1^1}X_n$$

and so $X_1 \in \mathrm{span}\{Y_1,X_2,\dots,X_n\}$. This set is linearly independent, else

$$Y_1 = \alpha_1^2X_2 + \cdots + \alpha_1^nX_n,$$

contradicting the assumption that $\alpha_1^1 \neq 0$. It follows that $\{Y_1,X_2,\dots,X_n\}$ is a basis for $\mathfrak{g}$.

Now consider $Y_2 \in \mathfrak{g}$. Using the newly formed basis,

$$Y_2 = \alpha_2^1 Y_1 + \alpha_2^2 X_2 + \cdots + \alpha_2^n X_n$$

where $\alpha_2^1, \ldots, \alpha_2^n \in \mathbb{R}$ and $\alpha_2^j \neq 0$ for at least one $j = 1, \ldots, n$. If $\alpha_2^j = 0$ for all $j > 1$, then $Y_2 = \alpha_2^1 Y_1$, contradicting the assumption that $\{Y_1, \ldots, Y_k\}$ form a linearly independent set, hence $\alpha_2^j \neq 0$ for at least one $j = 2, \ldots, n$. Assume without loss of generality that $\alpha_2^2 \neq 0$. Then

$$X_2 = -\frac{\alpha_2^1}{\alpha_2^2} Y_1 + \frac{1}{\alpha_2^2} Y_2 - \frac{\alpha_2^3}{\alpha_2^2} X_3 - \cdots - \frac{\alpha_2^n}{\alpha_2^2} X_n$$

and so $X_2 \in \mathrm{span}\{Y_1, Y_2, X_3 \ldots, X_n\}$. As before, this set is linearly independent and it follows that $\{Y_1, Y_2, X_3, \ldots, X_n\}$ is a basis for $\mathfrak{g}$.

This process can be continued inductively, forming the basis $\{Y_1, \ldots, Y_k, X_{k+1}, \ldots, X_n\}$ for $\mathfrak{g}$. This basis has the property that $\eta(Y_i, Y_j) = \pm\delta_{ij}$ for $i, j \in \{1, \ldots, k\}$ and $\eta(X_i, X_j) = \pm\delta_{ij}$ for $i, j \in \{k+1, \ldots, n\}$. In particular, $\eta(X_i, X_j) \neq 0$ for $i, j \in \{k+1, \ldots, n\}$, so an orthogonal basis for $\mathfrak{g}$ can be formed using the Graham-Schmidt Method by letting

$$Y_i := X_i - \sum_{j=1}^{i-1} \frac{\eta(X_i, Y_j)}{\eta(Y_j, Y_j)} Y_j \tag{20}$$

for $i = k+1, \ldots, n$. This provides a basis $\{Y_1, \ldots, Y_n\}$ for $\mathfrak{g}$ such that $\mathfrak{h} = \mathrm{span}\{Y_1, \ldots, Y_k\}$, $\mathfrak{m} = \mathrm{span}\{Y_{k+1}, \ldots, Y_n\}$, and $\eta(Y_i, Y_j) = \pm\delta_{ij}$ for $i, j \in \{1, \ldots, n\}$. Specifically, $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ and $\eta(Y_i, Y_j) = \pm\delta_{ij}$ for $i, j \in \{k+1, \ldots, n\}$ (i.e. $\eta|_{\mathfrak{m}}$ is non-degenerate).

It only remains to show $\mathfrak{m}$ is a reductive complement of $\mathfrak{h}$ in $\mathfrak{g}$, i.e. that $[\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m}$. First, note from the construction of the basis $\{Y_1, \ldots, Y_n\}$ that for any Y_i in the basis for $\mathfrak{h}$, $\eta(Y_i, Y_j) = 0$ for $j = k+1, \ldots, n$, or all Y_j in the basis for $\mathfrak{m}$. Thus any vector in $\mathfrak{h}$ is orthogonal to any vector not in $\mathfrak{h}$; hence

$$\mathfrak{m} = \mathrm{span}\{Y_{k+1}, \ldots, Y_n\} = \mathfrak{h}^\perp.$$

Second, the inner-product η is ad$\mathfrak{h}$-invariant, so from Definition 8.2, for any $X \in \mathfrak{h}$ and $Y, Z \in \mathfrak{g}$

$$\eta(X \cdot Y, Z) + \eta(Y, X \cdot Z) = \eta\Big(\mathrm{ad}(X)(Y), Z\Big) + \eta\Big(Y, \mathrm{ad}(X)(Z)\Big) = 0,$$

Proposition 8.5 then implies

$$\eta\big([X, Y], Z\big) + \big(Y, [X, Z]\big) = 0. \tag{21}$$

This equation holds for any $Y, Z \in \mathfrak{g}$, so let Y be any element in $\mathfrak{m}$ and Z any element in $\mathfrak{h}$. X and Z are both elements of the Lie subalgebra $\mathfrak{h}$, therefore the bracket $[X, Z]$ remains in $\mathfrak{h}$. The compliment of $\mathfrak{h}$ is orthogonal to $\mathfrak{h}$, so $Y \in \mathfrak{m}$ implies that $\eta\big(Y, [X, Z]\big) = 0$. In this case, Equation 21 becomes

$$\eta\big([X, Y], Z\big) = 0,$$

where Z is any element of $\mathfrak{h}$. By the non-degeneracy assumption on $\eta|_{\mathfrak{h}}$,

$$[X, Y] \in \mathfrak{h}^\perp = \mathfrak{m}.$$

Therefore if $X \in \mathfrak{h}$ and $Y \in \mathfrak{m}$, then $[X, Y] \in \mathfrak{m}$, as required. $\square$

10. The fundamental theorem of invariant metrics on a homogeneous space

The ideas of the preceding sections lead to the following theorem:

Theorem 10.1. *There is a one-to-one correspondence between invariant metrics γ on a homogeneous space M and invariant inner products η on a vector space $\mathfrak{g}/\mathfrak{h}$, where $(\mathfrak{g}, \mathfrak{h})$ is a suitably chosen Lie algebra pair. Furthermore, if $\mathfrak{h}$ admits a reductive complement $\mathfrak{m}$, then $\mathfrak{g}/\mathfrak{h}$ can be replaced with $\mathfrak{m}$ in the previous sentence.*

References

[1] W. M. Boothby, *An Introduction to Differentiable Manifolds and Riemannian Geometry*, San Diego, California: Academic Press, second ed., 1986.

[2] S. Kobayashi and K. Nomizu, *Foundations of differential geometry. Vol. II*, Interscience Tracts in Pure and Applied Mathematics, No. 15 Vol. II, Interscience Publishers John Wiley & Sons, Inc., New York-London-Sydney, 1969.

[3] H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers, and E. Herlt, *Exact solutions of Einstein's field equations*, Cambridge Monographs on Mathematical Physics, Cambridge University Press, Cambridge, second ed., 2003.

[4] F. W. Warner, *Foundations of Differentiable Manifolds and Lie Groups*, New York: Springer-Verlag, 1983.