Part I.

1. Let $R$ be a ring and $F : M \to N$ a homomorphism of $R$-modules. Prove that the following are equivalent.
   (a) $f$ is surjective.
   (b) $f_p : M_p \to N_p$ is surjective for each prime ideal $p$ of $R$.
   (c) $f_m : M_m \to N_m$ is surjective for each maximal ideal $m$ of $R$.

2. Let $p$ be a prime number. For $n \geq m$ let $f_{nm} : \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^m\mathbb{Z}$ be the canonical projection, i.e. $f_{nm}(a \mod p^n) = a \mod p^m$.
   (a) Show that $\{\mathbb{Z}/p^n\mathbb{Z}\}$ with homomorphisms $f_{nm}$ forms an inverse system of commutative rings. Let $\mathbb{Z}_p$ denote $\lim_{\leftarrow} \mathbb{Z}/p^n\mathbb{Z}
   (b) Find the canonical image of $\mathbb{Z}$ in $\mathbb{Z}_p$ and show that $\mathbb{Z}_p$ is an integral domain.
   (c) Show that $\mathbb{Z}_p$ is a local ring and an principal ideal domain.
   The ring $\mathbb{Z}_p$ is called the ring of $p$-adic integers.

3. Let $p$ be a prime and let $R$ be the set of formal power series in $p$:
   $$ R = \left\{ \sum_{n=0}^{\infty} a_n p^n : a_n = 0, 1, \ldots, p-1 \right\}. $$
   (a) Show that $R$ is a commutative ring under the addition and multiplication of power series (do show that multiplication makes sense!).
   (b) Show that $\mathbb{Z}_p$ is naturally isomorphic to $R$.

Bonus Let $\mathbb{N}$ be the set of positive integers ordered by divisibility. Observe that
$$ \{\mathbb{Z}/n\mathbb{Z}\}_{n \in \mathbb{N}} $$
forms an inverse system of commutative rings with the canonical homomorphisms $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ for $m \mid n$. Let $\hat{\mathbb{Z}} = \lim_{\leftarrow} \mathbb{Z}/n\mathbb{Z}$. Show that
$$ \hat{\mathbb{Z}} \cong \prod_{p \text{ prime}} \mathbb{Z}_p. $$

Part II. From Atiyah-Macdonald

Chapter 5: 1, 3, 4, 8, 9