

## Problems inspired by Lecture 1: The Probabilistic Method

1. Let  $n(k)$  be the smallest  $n$  such that some tournament with  $n$  players has the property  $S_k$  (from page 5). Spencer claims it is “easy to show” that  $n(k) > 2^{k-1}$ . Is it?
2. For a given  $k, n$ , consider the cube  $[k]^n$  — ordered  $n$ -tuples of numbers from  $1, \dots, k$ . A line in  $[k]^n$  is a set of  $k$  points from  $[k]^n$ ,  $a_1, \dots, a_k$ , with a nonempty collection of indices  $\lambda \subseteq [n]$  such that  $a_i$  has  $t^{\text{th}}$  coordinate  $i$  for  $t \in \lambda$ , and all  $a_i$  agree on coordinates outside of  $\lambda$ . Example: when  $k = 3, n = 2$ , these are lines in the sense of tic-tac-toe.

The Hales-Jewett theorem states that, for each  $k$ , there is some  $n = HJ(k)$  so that every 2-coloring of  $[k]^n$  contains a monochromatic line. Use the probabilistic method to get a lower bound on  $HJ(k)$ .

3. Let  $n \geq 4$ ,  $H$  be an  $n$ -uniform hypergraph with at most  $\frac{4^{n-1}}{3^n}$  edges. Prove that there is a 4-coloring of the vertices of  $H$  so that each edge represents all four colors.
4. Prove that there is some constant  $c$  so that, for any  $n$ , the following holds: If  $A$  is an  $n \times n$  matrix with distinct entries (may as well be  $1, \dots, n^2$ ), then there is some permutation of the rows of  $A$  so that no column contains an increasing subsequence of length  $\geq c\sqrt{n}$ .