

PROBLEMS FROM LECTURE 4

1. (Spencer p. 232, #1) Let \mathcal{A} be a family of n subsets of $\Omega = \{1, \dots, m\}$ with m even. Let $\chi(i)$, $1 \leq i \leq m/2$, be independent and uniform in $\{\pm 1\}$, and set $\chi(i + m/2) = -\chi(i)$ for $1 \leq i \leq m/2$. Using this notion of random coloring, show $\text{disc}(\mathcal{A}) \leq \sqrt{m \ln(2n)}$. Show that this can be improved even further by splitting Ω randomly into $m/2$ disjoint pairs.

2. (Spencer p. 272, #5) Let $X_i = \pm 1$, $1 \leq i \leq n$, be uniform and independent and let $S_n = \sum_{i=1}^n X_i$. Let $0 \leq p \leq 1/2$. Then

$$P(S_n \geq (1 - 2p)n) \leq 2^{H(p)n - n},$$

where $H(p) = -p \log_2 p - (1 - p) \log_2(1 - p)$, the binary entropy function. (Spencer says this may be done by computing precisely the Chernoff bound $\min_{\lambda \geq 0} \mathbb{E}[e^{\lambda S_n}] e^{-\lambda(1-2p)n}$.)

3. (The Liar Game, Spencer p. 258) This is proven in Spencer, but the ideas are similar to the lecture so it may be fun to give it a go ourselves. Here's the setup: we have n chips. Let $(x_0, x_1, \dots, x_k) \in \mathbb{Z}^k$, where x_i denotes the number of chips at position i at the beginning of the game. On each round, Paul selects a set S of chips on the board. Carole then either moves all chips in S to the left by one, or moves all chips in S^c to the left by one. If a chip's position becomes negative as a result, it is removed from the game. This is done for q rounds. If there are at least 2 chips on the board at the end, Carole wins; otherwise, Paul wins.

Let $B(q, j) = 2^{-1} \sum_{i=0}^j \binom{q}{i}$, the binomial distribution with parameters q, j . Prove that if $\sum_{i=0}^k x_i B(q, i) > 1$, then Carole wins the game. Give an explicit strategy by which Carole can win.