Math 241A: Functional Analysis

Homework 1

due in class Monday, October 27th

Unless otherwise specified, all vector spaces are considered over $\mathbb{C}$.

1. Let $\mathcal{H}$ be a Hilbert space and $\alpha \in \mathbb{C}$ such that $\alpha^2 \neq 1$ and $\alpha^N = 1$, for an integer $N > 2$. Prove that $\langle \xi, \eta \rangle = \frac{1}{N} \sum_{n=1}^{N} \|\xi + \alpha^n \eta\|^2 \alpha^n$, for all $\xi, \eta \in \mathcal{H}$.

2. Let $\mathcal{H}$ be a Hilbert space such that the *closed unit ball* $\{\xi \in \mathcal{H} \mid \|\xi\| \leqslant 1\}$ is compact. Prove that $\mathcal{H}$ is finite dimensional.

3. Let $\mathcal{H}$ be an infinite dimensional Hilbert space. Prove that there does not exist a countable set $S \subset \mathcal{H}$ which spans $\mathcal{H}$ (as a vector space).

4. Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert spaces. Prove that $\dim(\mathcal{H}) = \dim(\mathcal{K})$ if and only if there a linear surjective map $U : \mathcal{H} \to \mathcal{K}$ such that $\|U(\xi)\| = \|\xi\|$, for all $\xi \in \mathcal{H}$ (such a map U is called a *unitary*).

5. Let $(X, \|.\|)$ be a normed vector space over $\mathbb{R}$ which satisfies the *parallelogram law*: $\|\xi + \eta\|^2 + \|\xi - \eta\|^2 = 2(\|\xi\|^2 + \|\eta\|^2)$, for all $\xi, \eta \in X$. Prove that there is an inner product $\langle ., . \rangle : X \times X \to \mathbb{R}$ such that $\|\xi\|^2 = \langle \xi, \xi \rangle$, for all $\xi \in X$.

6. Let X and Y be normed spaces and $T : X \to Y$ be a linear map. Prove that T is continuous if and only if there is a constant $c > 0$ such that $\|T(\xi)\| \leqslant c\|\xi\|$, for all $\xi \in X$.

7. Let (X, μ) be a σ-finite measure space, $\phi \in L^\infty(X, \mu)$ and $1 \leqslant p \leqslant +\infty$. Define $M_\phi : L^p(X, \mu) \to L^p(X, \mu)$ by letting $M_\phi(f) = \phi f$, for all $f \in L^p(X, \mu)$. Prove that $M_\phi \in B(L^p(X, \mu))$ and $\|M_\phi\| = \|\phi\|_\infty$ (i.e. the *essential supremum* of ϕ).

8. Let X and Y be Banach spaces, and $T \in B(X, Y)$. Prove that there is a constant $c > 0$ such that $\|T(\xi)\| \geqslant c\|\xi\|$, for all $\xi \in X$, if and only if $\ker(T) = \{0\}$ and $\mathrm{ran}(T)$ is closed.

9. Let $1 \leqslant p < \infty$ and let ℓ^p denote the Banach space of ℓ^p*-summable* functions $f : \mathbb{N} \to \mathbb{C}$, i.e. such that $\|f\|_p := (\sum_{n \in \mathbb{N}} |f(n)|^p)^{\frac{1}{p}}$ is finite. Let $T : \ell^p \to \ell^p$ be a linear map. Assume that there is an infinite matrix $(\alpha_{i,j})_{i,j=1}^{\infty}$ of complex numbers such that $(Tf)(i) = \sum_{j=1}^{\infty} \alpha_{i,j} f(j)$, for all $f \in \ell^p$ and every $i \geqslant 1$. Prove that $T \in B(\ell^p)$.

10. Let X be a separable normed space. Prove that the unit ball of X^* ($\{\phi \in X^* \mid \|\phi\| \leqslant 1\}$) is a separable metrizable space with the weak*-topology.

11. Let $X = L^\infty([0, 1], \lambda)$, where λ is the Lebesgue measure on $[0, 1]$. For $n \geqslant 1$, let $\phi_n \in X^*$ be given by $\phi_n(f) = \frac{1}{n} \int_0^{\frac{1}{n}} f \, \mathrm{d}\lambda$, for any $f \in L^\infty([0, 1], \lambda)$. Note that $\|\phi_n\| \leqslant 1$, for all $n \geqslant 1$, and let $\phi \in X^*$ be any point in the weak*-closure of the sequence $\{\phi_n\}_{n \geqslant 1}$. Prove that ϕ is not of the form ϕ_g, for any $g \in L^1([0, 1], \lambda)$, where $\phi_g(f) = \int_0^1 fg \, d\lambda$.