Math 241A: Functional Analysis

Homework 1

due in class Monday, October 27th

Unless otherwise specified, all vector spaces are considered over \( \mathbb{C} \).

1. Let \( \mathcal{H} \) be a Hilbert space and \( \alpha \in \mathbb{C} \) such that \( \alpha^2 \neq 1 \) and \( \alpha^N = 1 \), for an integer \( N > 2 \). Prove that \( \langle \xi, \eta \rangle = \frac{1}{N} \sum_{n=1}^{N} \| \xi + \alpha^n \eta \| \alpha^n \), for all \( \xi, \eta \in \mathcal{H} \).

2. Let \( \mathcal{H} \) be a Hilbert space such that the closed unit ball \( \{ \xi \in \mathcal{H} \mid \| \xi \| \leq 1 \} \) is compact. Prove that \( \mathcal{H} \) is finite dimensional.

3. Let \( \mathcal{H} \) be an infinite dimensional Hilbert space. Prove that there does not exist a countable set \( S \subset \mathcal{H} \) which spans \( \mathcal{H} \) (as a vector space).

4. Let \( \mathcal{H} \) and \( \mathcal{K} \) be two Hilbert spaces. Prove that \( \dim(\mathcal{H}) = \dim(\mathcal{K}) \) if and only if there is a linear surjective map \( U : \mathcal{H} \to \mathcal{K} \) such that \( \| U(\xi) \| = \| \xi \| \), for all \( \xi \in \mathcal{H} \) (such a map \( U \) is called a unitary).

5. Let \( (X, \| \cdot \|) \) be a normed vector space over \( \mathbb{R} \) which satisfies the parallellogram law: \( \| \xi + \eta \|^2 + \| \xi - \eta \|^2 = 2(\| \xi \|^2 + \| \eta \|^2) \), for all \( \xi, \eta \in X \). Prove that there is an inner product \( \langle \cdot, \cdot \rangle : X \times X \to \mathbb{R} \) such that \( \| \xi \|^2 = \langle \xi, \xi \rangle \), for all \( \xi \in X \).

6. Let \( X \) and \( Y \) be normed spaces and \( T : X \to Y \) a linear map. Prove that \( T \) is continuous if and only if there is a constant \( c > 0 \) such that \( \| T(\xi) \| \leq c \| \xi \| \), for all \( \xi \in X \).

7. Let \( (X, \mu) \) be a \( \sigma \)-finite measure space, \( \phi \in L^\infty(X, \mu) \) and \( 1 \leq p \leq +\infty \). Define \( M_\phi : L^p(X, \mu) \to L^p(X, \mu) \) by letting \( M_\phi(f) = \phi f \), for all \( f \in L^p(X, \mu) \). Prove that \( M_\phi \in B(L^p(X, \mu)) \) and \( \| M_\phi \| = \| \phi \|_\infty \) (i.e. the essential supremum of \( \phi \)).

8. Let \( X \) and \( Y \) be Banach spaces, and \( T \in B(X, Y) \). Prove that there is a constant \( c > 0 \) such that \( \| T(\xi) \| \geq c \| \xi \| \), for all \( \xi \in X \), if and only if \( \ker(T) = \{ 0 \} \) and \( \text{ran}(T) \) is closed.

9. Let \( 1 \leq p < \infty \) and let \( \ell^p \) denote the Banach space of \( \ell^p \)-summable functions \( f : \mathbb{N} \to \mathbb{C} \), i.e. such that \( \| f \|_p := (\sum_{n \in \mathbb{N}} |f(n)|^p)^{\frac{1}{p}} \) is finite. Let \( T : \ell^p \to \ell^p \) be a linear map. Assume that there is an infinite matrix \( (\alpha_{ij})_{i,j=1}^\infty \) of complex numbers such that \( (Tf)(i) = \sum_{j=1}^\infty \alpha_{ij} f(j) \), for all \( f \in \ell^p \) and every \( i \geq 1 \). Prove that \( T \in B(\ell^p) \).

10. Let \( X \) be a separable normed space. Prove that the unit ball of \( X^* \) (\( \{ \phi \in X^* \mid \| \phi \| \leq 1 \} \)) is a separable metrizable space with the weak*-topology.

11. Let \( X = L^\infty([0, 1], \lambda) \), where \( \lambda \) is the Lebesgue measure on \([0, 1] \). For \( n \geq 1 \), let \( \phi_n \in X^* \) be given by \( \phi_n(f) = \frac{1}{n} \int_0^1 f d\lambda \), for any \( f \in L^\infty([0, 1], \lambda) \). Note that \( \| \phi_n \| \leq 1 \), for all \( n \geq 1 \), and let \( \phi \in X^* \) be any point in the weak*-closure of the sequence \( \{ \phi_n \}_{n \geq 1} \). Prove that \( \phi \) is not of the form \( \phi_g \), for any \( g \in L^1([0, 1], \lambda) \), where \( \phi_g(f) = \int_0^1 fg d\lambda \).