Math 247A Homework 1

Instructions: *attempt at least* **5** *problems and turn them in during class by* **January 30**.

In the problems below, H denotes a complex Hilbert space.

Problem 0.1. Let $T \in \mathbb{B}(H)$ and denote by $T^* \in \mathbb{B}(H)$ its adjoint.

Prove that $\|T^*\| = \|T\|$ and $\|T^*T\| = \|T\|^2$.

Problem 0.2. Let $(T_n) \subset \mathbb{B}(H)$ be a sequence of unitary operators. Assume that $T_n \to T$ (WOT), where $T \in \mathbb{B}(H)$ is a unitary operator.

Prove that $T_n \to T$ (SOT).

Problem 0.3. Assume that H is infinite dimensional and separable ($\Leftrightarrow H$ has a countable infinite orthonormal basis).

Give an example of a sequence of unitaries (T_n) which converge in the WOT but not the SOT.

Problem 0.4. Let $\mathcal{B} = \{T \in \mathbb{B}(H) \mid \|T\| \leq 1\}$ denote the closed unit ball of $\mathbb{B}(H)$.

Prove that $\mathcal{B}$ is compact in the WOT.

Problem 0.5. Let $B \subset \mathbb{B}(H)$ be a set of operators such that $T^* \in B$, for every $T \in B$.

Prove that $B' := \{T \in \mathbb{B}(H) \mid TS = ST, \text{ for every } S \in B\}$ is a von Neumann algebra.

Problem 0.6. Let I be a set. Define a $*$-homomorphism $\pi : \ell^\infty(I) \to \mathbb{B}(\ell^2(I))$ by the formula $\pi(f)(g)(i) = f(i)g(i)$, for all $i \in I$, $f \in \ell^\infty(I)$ and $g \in \ell^2(I)$.

Prove that $\pi(\ell^\infty(I))' = \pi(\ell^\infty(I))$.

Problem 0.7. Let Γ be a finite abelian group and put $n = |\Gamma|$.

Prove that $L(\Gamma)$ is $*$-isomorphic to $\ell^\infty(\{1, 2, ..., n\})$.

Problem 0.8. An operator $T \in \mathbb{B}(H)$ is called positive if $T = T^*$ and $\sigma(T) \subset [0, \infty)$.

Prove that an operator $T \in \mathbb{B}(H)$ is positive if and only if $\langle T\xi, \xi \rangle \geqslant 0$, for all $\xi \in H$.

Problem 0.9. Let $A \subset \mathbb{B}(H)$ be a C*-algebra.

(1) Prove that every self-adjoint element $a \in A$ is a linear combination of two unitary elements of A.
(2) Prove that every element $a \in A$ is a linear combination of four unitary elements of A.

Problem 0.10. (the Cauchy-Schwarz inequality) Let A be a C*-algebra and $\varphi : A \to \mathbb{C}$ be a linear functional. Assume that φ is positive, that is, $\varphi(x^*x) \geq 0$, for all $x \in A$.

Prove that $|\varphi(y^*x)|^2 \leqslant \varphi(x^*x)\varphi(y^*y)$, for all $x, y \in A$.