Math 241A: Functional Analysis

Homework 2

due in class Monday, November 17th

1. Let m be the Haar measure of a compact group G.

- Prove that $m(V) > 0$, for any non-empty open subset $V \subset G$.
- Prove that if $H < G$ is an open subgroup, then the index $[G : H]$ is finite.
- Prove that $m(\Delta^{-1}) = m(\Delta)$, for any Borel subset $\Delta \subset G$, where $\Delta^{-1} = \{g^{-1} | g \in \Delta\}$.

2. Let G be a topological group and let $H \subset G$ be the largest connected subset of G which contains the identity element. Prove that H is a normal subgroup of G (*Hint:* show first that if $A, B \subset G$ are connected sets, then AB and A^{-1} are connected).

3. A *mean* on $\ell^\infty(\mathbb{Z})$ is a linear functional $\phi : \ell^\infty(\mathbb{Z}) \to \mathbb{C}$ such that $\phi(1) = 1$ and $\phi(f) \geqslant 0$, for any $f \in \ell^\infty(\mathbb{Z})$ with $f \geqslant 0$. Let $\mathcal{M}$ be the set of means of $\ell^\infty(\mathbb{Z})$.

- Prove that $\mathcal{M}$ is a closed convex subset of the unit ball of $\ell^\infty(\mathbb{Z})$.
- Prove that there is an *invariant* mean $\phi \in \mathcal{M}$, i.e. such that $\phi(\sigma_n(f)) = \phi(f)$, for any $f \in \ell^\infty(\mathbb{Z})$ and $n \in \mathbb{Z}$. Here, $\sigma_n(f) \in \ell^\infty(\mathbb{Z})$ is given by $\sigma_n(f)(x) = f(x-n)$, for $x \in \mathbb{Z}$.
- Prove that if ϕ is an invariant mean and $A \subset \mathbb{Z}$ is a finite set, then $\phi(1_A) = 0$.
- Prove that if ϕ is an invariant mean, then $\phi : \ell^\infty(\mathbb{Z}) \to \mathbb{C}$ is not weak*-continuous (where we use the identification $\ell^\infty(\mathbb{Z}) = \ell^1(\mathbb{Z})^*$).

4. Let $(X, \|.\|)$ be a Banach space. A norm $\|.\|'$ on X is *equivalent* to $\|.\|$ if there are constants $c, C > 0$ such that $c\|x\| \leqslant \|x\|' \leqslant C\|x\|$, for every $x \in X$. Let $T : X \to X$ be an isomorphism such that $\sup_{n \geqslant 1} \|T^n\| < +\infty$. Prove that there exists an equivalent norm $\|.\|'$ such that T is isometric with respect to $\|.\|'$, i.e. we have $\|T(x)\|' = \|x\|'$, for every $x \in X$.

5. Let $I = [0, 1]$ and $\phi : I \to I$ be given by $\phi(x) = x^2$. Find all probability measures on I that are ϕ-invariant.

6. Let G be a topological group such that $G = \cup_{n \geqslant 1} G_n$, where $G_n \subset G$ is a compact subgroup and $G_n \subset G_{n+1}$, for all $n \geqslant 1$. Assume that G acts continuously on a compact metric space X. Prove that there is a G-invariant probability measure on X.

7. Let X be a compact metric space and denote by $\mathcal{M}(X)$ the Borel probability measures on X. Prove that the exterior points of $\mathcal{M}(X)$ are precisely the Dirac measures $\{\delta_x\}_{x \in X}$. Recall that if $A \subset X$, then $\delta_x(A) = 1$, if $x \in A$, and $\delta_x(A) = 0$, if $x \notin A$.

8. Let G be a compact group which acts continuously on a compact metric space (X, d). Prove that there is a metric d' on X which defines the same topology and is G-invariant, i.e. satisfies $d'(gx, gy) = d'(x, y)$, for all $g \in G$ and every $x, y \in X$.