Math 247A Homework 2

Instructions: *this homework is* **optional** *and can be turned in during class by* **March 13**.

In the problems below, H denotes a complex Hilbert space.

Problem 0.1. Let $M \subset \mathbb{B}(H)$ be a unital von Neumann algebra.

(1) Prove that if $M \neq \{\alpha \cdot 1 \mid \alpha \in \mathbb{C}\}$, then there exists a projection $p \in M \setminus \{0, 1\}$.
(2) Prove that a projection $p \in M$ is minimal if and only if $pMp = \{\alpha \cdot p \mid \alpha \in \mathbb{C}\}$.
(A projection $p \in M$ is minimal if any projection $q \in M$ such that $q \leq p$ is equal to 0 or p.)

Problem 0.2. Prove that $\mathbb{B}(H)$ is a factor of type I.

Problem 0.3. Let M be a type II_1 factor and $\tau : M \to \mathbb{C}$ be a faithful normal tracial state.

Prove that two projections $p, q \in M$ are equivalent if and only if $\tau(p) = \tau(q)$.

Problem 0.4. Let M be a type II_1 factor and $\tau : M \to \mathbb{C}$ be a faithful normal tracial state.

(1) Prove that there exists a projection $p \in M$ such that $\tau(p) = 1/2$.
(2) Prove that there exists an injective unital $*$-homomorphism $\rho : \mathbb{M}_2(\mathbb{C}) \to M$.
(3) Prove that there exists a unital $*$-homomorphism $\pi : R \to M$.

Problem 0.5. Let Γ be an infinite group and (Y, ν) be a non-trivial standard probability space. Define $(X, \mu) = (Y^\Gamma, \nu^{\otimes \Gamma})$. Consider the Bernoulli action $\Gamma \curvearrowright (X, \mu)$ given by

$$g \cdot x = (x_{g^{-1}h})_{h \in \Gamma}, \text{ for every } g \in \Gamma \text{ and } x = (x_h)_{h \in \Gamma} \in X.$$

Prove that for any measurable sets $Y, Z \subset X$ we have that $\lim_{g \to \infty} \mu(gY \cap Z) = \mu(Y)\mu(Z)$.

Problem 0.6. Let Γ_1 and Γ_2 be any countable groups such that $|\Gamma_1| > 1$ and $|\Gamma_2| > 2$.

Prove that the free product group $\Gamma = \Gamma_1 * \Gamma_2$ is icc.

Problem 0.7. Let Γ_1 and Γ_2 be any countable groups such that $|\Gamma_1| > 1$ and $|\Gamma_2| > 2$.

Prove that the free product group $\Gamma = \Gamma_1 * \Gamma_2$ is not amenable.

Problem 0.8. Let Γ be an icc countable group and $\Gamma \curvearrowright (X, \mu)$ be a pmp action.

Prove that if the action $\Gamma \curvearrowright (X, \mu)$ is ergodic, then $L^\infty(X) \rtimes \Gamma$ is a factor.