Math 241A: Functional Analysis

Homework 3

due in class Wednesday, December 10th

1. Let $\mathcal{H}$ be a Hilbert space and $T : \mathcal{H} \to \mathcal{H}$ be a compact operator. Prove that there exists a sequence $T_n : \mathcal{H} \to \mathcal{H}$ of *finite rank* operators (i.e. such that $\dim(T_n(\mathcal{H})) < +\infty$) such that $\lim_{n\to\infty} \|T_n - T\| = 0$.

2. Let $\mathcal{H}$ be a Hilbert space and $T : \mathcal{H} \to \mathcal{H}$ be a bounded operator. A sequence $\{\xi_n\}_{n\geqslant 1}$ is said to converge weakly to 0 if $\langle \xi_n, \eta \rangle \to 0$, for every $\eta \in \mathcal{H}$. Prove that T is compact if and only if $\|T(\xi_n)\| \to 0$, for any sequence $\{\xi_n\}_{n\geqslant 1}$ in $\mathcal{H}$ that converges weakly to 0.

3. Let G be a compact abelian group and $n \geqslant 1$ be an integer. Let $\pi : G \to \mathcal{U}(\mathbb{C}^n)$ be an irreducible unitary representation. Prove that $n = 1$. (*Hint:* If $g \in G$, then the self-adjoint operator $\pi(g) + \pi(g)^* = \pi(g) + \pi(g^{-1})$ commutes with $\pi(G)$. Hence, any eigenspace of $\pi(g) + \pi(g)^*$ is $\pi(G)$-invariant. Deduce that $\pi(g) + \pi(g)^* \in \mathbb{C}1$. Similarly, show that $i(\pi(g) - \pi(g)^*) \in \mathbb{C}1$).

4. Let $\mathcal{H}$ be a Hilbert space and $T \in B(\mathcal{H})$. Prove that T can be written as a linear combination of at most 4 unitary operators.

5. Let $\mathcal{H}$ be a Hilbert space and $T \in B(\mathcal{H})$. Prove that T is normal (i.e. $T^*T = TT^*$) if and only if $\|T(\xi)\| = \|T^*(\xi)\|$, for all $\xi \in \mathcal{H}$.

6. Let $\mathcal{H}$ be a Hilbert space and $T \in B(\mathcal{H})$ such that $T^* = T$. Denote by $\mathcal{F}$ the family of operators of the form $S = \sum_{i=1}^{n} c_i P_i$, where $c_1, ..., c_n \in \mathbb{C}$ and $P_1, ..., P_n \in B(\mathcal{H})$ are projections such that $P_i(\mathcal{H}) \perp P_j(\mathcal{H})$, for all $1 \leqslant i < j \leqslant n$, and $P_i T = T P_i$, for all $1 \leqslant i \leqslant n$. Prove that there exists a sequence $T_m \in \mathcal{F}$ such that $\lim_{m\to\infty} \|T_m - T\| = 0$.

7. Let $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ and $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Define $\phi : S^1 \to S^1$ by letting $\phi(z) = e^{2\pi i \alpha} z$ (the rotation by angle $2\pi\alpha$). Note that ϕ leaves the Lebesgue measure λ of S^1 invariant.

- Prove that ϕ is *ergodic*: if a Borel set $A \subset S^1$ satisfies $\phi(A) = A$, then either $\lambda(A) = 0$ or $\lambda(S^1 \setminus A) = 0$ (i.e. A is null or conull).
- Prove that ϕ is not mixing. (Recall that ϕ is *mixing* if $\lim_{n\to\infty} \lambda(\phi^n(A) \cap B) = \lambda(A)\lambda(B)$, for any Borel sets $A, B \subset S^1$).