Math 240A, make-up lecture, Monday, October 23

Terminology. Let \((X, \mathcal{M})\) be a measurable space. A function \(f : X \to \mathbb{R}\) is \(\mathcal{M}\)-measurable if it is \((\mathcal{M}, \mathcal{B}_\mathbb{R})\)-measurable: \(f^{-1}(E) \in \mathcal{M}, \forall E \in \mathcal{B}_\mathbb{R}\).

Let \(\overline{\mathbb{R}} = [-\infty, +\infty]\) the extended real numbers
\[ \mathcal{B}_{\overline{\mathbb{R}}} = \{E \subset \overline{\mathbb{R}} \mid \exists \mathcal{A} \in \mathcal{B}_\mathbb{R}, E \cap \mathcal{A} \in \mathcal{B}_\mathbb{R}\} \]
be the \(\sigma\)-algebra generated by \((a, +\infty]\) or \([-\infty, a)\), \(a \in \mathbb{R}\).

A function \(f : X \to \overline{\mathbb{R}}\) is \(\mathcal{M}\)-measurable if it is \((\mathcal{M}, \mathcal{B}_{\overline{\mathbb{R}}})\)-measurable.

**Proposition 1.** Let \((X, \mathcal{M})\) be a measurable space and \(f_i : X \to \mathbb{R}, i \geq 1\), be \(\mathcal{M}\)-measurable functions.

Then
\[
\begin{align*}
    g_1(x) &= \sup_{i \geq 1} f_i(x) \\
g_2(x) &= \inf_{i \geq 1} f_i(x) \\
g_3(x) &= \limsup_{i \to \infty} f_i(x) \\
g_4(x) &= \liminf_{i \to \infty} f_i(x)
\end{align*}
\]

are \(\mathcal{M}\)-measurable.

Moreover, if \(f(x) = \lim_{i \to \infty} f_i(x)\) exists for every \(x \in X\),
then \(f\) is \(\mathcal{M}\)-measurable.

**Proof.** Since \(g_1^{-1}((a_1, \infty]) = \bigcup_{i \geq 1} f_i^{-1}((a_1, \infty]) \in \mathcal{M}\)
and \(g_2^{-1}((a_1, \infty]) = \bigcup_{n \geq 1} \bigcap_{i \geq 1} f_i^{-1}((a_1 + \frac{1}{n}, \infty]) \in \mathcal{M}\),
g_1 and g_2 are \(\mathcal{M}\)-measurable.

Thus, \(g_3(x) = \inf_{i \geq 1} \left( \sup_{k \geq i} f_k(x) \right)\)
and \(g_4(x) = \sup_{i \geq 1} \left( \inf_{k \geq i} f_k(x) \right)\) are \(\mathcal{M}\)-measurable.

Moreover, if \(f\) exists, then \(f = g_1 = g_2\) is \(\mathcal{M}\)-measurable.
Corollary 2. If \( f, g : X \to \mathbb{R} \) are measurable, then

\[
\begin{align*}
\text{max} (f, g) &:= \max (f(x), g(x)) \\
\text{min} (f, g) &:= \min (f(x), g(x)) \\
f^+(x) &= \max (f(x), 0) \geq 0 \\
f^-(x) &= -\min (f(x), 0) \geq 0 \quad \text{are all measurable.}
\end{align*}
\]

Notation \( f^+ \) and \( f^- \) are called the positive and negative parts of \( f \). Note that \( f = f^+ - f^- \).

Remark. If \( f : X \to \mathbb{C} \) is measurable, then

\[ f = (\text{sgn} f) \cdot |f| \quad \text{is called the polar decomposition of} \; f. \]

Since \( \text{sgn} : \mathbb{C} \to \mathbb{C} \) is Borel measurable (Exercise)

and \( |f| : \mathbb{C} \to \mathbb{C} \) is continuous,

we have that \( \text{sgn} f = \text{sgn} \; \text{of} \) and \( |f| = | \text{of} \) are measurable.

Goal. \((X, M, \mu)\) measure space.

Define \( \int_X f \, d\mu \), for \( f \geq 0 \) measurable.

1) first for characteristic functions \((\int_X 1_E \, d\mu = \mu(E) )\)

and their linear combinations.

2) second, approximate any \( f \geq 0 \) by simple functions \( f_n \)

and define \( \int_X f \, d\mu = \lim_{n \to \infty} \int_X f_n \, d\mu \).

3) extend by linearity for complex-valued functions:

\[ f : X \to \mathbb{C} \quad f = f_1 + if_2 = (f_1^+ - f_1^-) + i(f_2^+ - f_2^-) \]
Let \((X, \mathcal{M})\) be a measurable space.

**Definition:** A simple function \(f: X \to \mathbb{C}\) is a finite linear combination of characteristic functions of sets in \(\mathcal{M}\), i.e. \(f = \sum_{i=1}^{m} c_i \chi_{F_i}\), for some \(c_1, \ldots, c_m \in \mathbb{C}, F_1, \ldots, F_m \in \mathcal{M}\).

**Remarks:**
1) \(f\) is a simple function \(\iff\) \(f\) is measurable and \(\mathbb{R}(X) < \mathbb{C}\) is finite.
2) If \(f\) is a simple function and \(f(\mathbb{Z}) = \{2, 4, \ldots, 2^n\}\), then
   \[
   f = \sum_{i=1}^{n} 2^i \chi_{E_i}, \text{ where } E_i = f^{-1}(2^i),
   \]
   is called the standard representation.

**Theorem:** Let \((X, \mathcal{M})\) be a measurable space, and \(f: X \to [0, +\infty]\) be a measurable function.

Then \(\exists\) an increasing sequence of simple functions \(\{\psi_n\}\), such that

1) \(\psi_n \not\to f\) pointwise, and
2) \(\psi_n \to f\) uniformly on any set on which \(f\) is bounded.

**Proof:** Let \(\psi_n(x) = \begin{cases} \frac{i-1}{2^n}, & \text{if } f(x) \in \left[\frac{i-1}{2^n}, \frac{i}{2^n}\right) \\ \frac{i}{2^n}, & \text{if } f(x) \geq n \end{cases}\) for some \(i = 1, 2, \ldots, 2^n\).

If \(f(x) < n\), then \(f(x) \in \left[\frac{i-1}{2^n}, \frac{i}{2^n}\right) = \left[\frac{2(i-1)}{2^{n+1}}, \frac{2i-1}{2^{n+1}}\right) \cup \left[\frac{2i-1}{2^{n+1}}, \frac{2i}{2^{n+1}}\right)\),

which shows that \(\psi_n(x) \leq \psi_{n+1}(x)\).

If \(f(x) \geq n\), then \(f(x) \geq \frac{n2^{n+1}}{2^{n+1}} \Rightarrow \psi_{n+1}(x) \geq n = \psi_n(x)\).

This altogether proves that \(\psi_n \leq \psi_{n+1}\).
If $f(x) = +\infty$, then $y_n(x) = n \not\rightarrow f(x)$.
If $f(x) \in [0, \infty)$, let $N \in \mathbb{N}$ s.t. $f(x) < N$.
Then $\forall n \geq N$, $0 \leq f(x) < n \Rightarrow 0 \leq f(x) - y_n(x) < \frac{1}{2^n}$, again proving that $y_n(x) \not\rightarrow f(x)$.

For (2), let $E \subset M$ such that $\forall C > 0$ satisfying

$0 \leq f(x) \leq C$, $\forall x \in E$.

Then $\forall n \geq C$, we have $0 \leq f(x) - y_n(x) < \frac{1}{2^n}$, $\forall x \in E$,
proving that $y_n \rightarrow f$ uniformly on $E$.

**Corollary:** If $f: X \rightarrow C$ is measurable, then

1. a sequence of simple functions $\{\Phi_n\}_{n \geq 1}$ such that
   1. $\{\Phi_n\}$ is an increasing sequence
   2. $\Phi_n \rightarrow f$ pointwise.
   3. $\Phi_n \rightarrow f$ uniformly on any set on which $f$ is bounded.

**Proof** Let $f = g + i h$

$= (g^+ - g^-) + i (h^+ - h^-)$, where $g^+, h^+: X \rightarrow [0, \infty)$

are measurable.

Let $0 \leq y_n^+ \leq g^+$ and $0 \leq y_n^- \leq h^+$ be the
sequences of simple functions provided by Thm (3).

Then (2) and (3) are clear for

$\Phi_n = (y_n^+ - y_n^-) + i (y_n^+ - y_n^-)$

(1) is also easy to check since

$|\Phi_n| = \sqrt{(y_n^+ + y_n^-)^2 + (y_n^+ + y_n^-)^2} \leq \sqrt{(g^1 + g^1)^2 + (h^1 + h)^2}$

$= |f|$.
Proposition 5 (exercise)
Assume \((X, \mathcal{M}, \mu)\) is a measure space, with \(\mu\) complete.

Then 1) \(f\) measurable and \(g=f\ \mu\text{-a.e.} \implies g\) measurable
2) \(f_n\) measurable, \(\forall n\), and \(f_n \rightarrow f\ \mu\text{-a.e.},\)
then \(f\) is measurable.

Proposition 6 Let \((X, \mathcal{M}, \mu)\) be a measure space, and \((X, \overline{\mathcal{M}}, \overline{\mu})\) be its completion.

If \(f: X \rightarrow \mathbb{R}\) is \(\overline{\mu}\)-measurable, then \(\exists g: X \rightarrow \mathbb{R}\)
\(\overline{\mu}\)-measurable such that \(g=f\ \overline{\mu}\text{-a.e.}\)

**Proof** For \(x \in \mathbb{Q}\), let \(E_x = f^{-1}((-\infty, x)) \in \overline{\mathcal{M}}.\)

By the construction of \(\overline{\mathcal{M}} \implies \exists E_x \in \mathcal{M}\) such that
\(E_x \subset F_x\) and \(\overline{\mu}(F_x \setminus E_x) = 0.\)

Moreover, \(F_x \setminus E_x \subset N_x \in \mathcal{M}\) with \(\mu(N_x) = 0.\)

Let \(N = \bigcup_{x \in \mathbb{Q}} N_x \in \mathcal{M}.\) Then \(\mu(N) = 0.\)

We define \(\tilde{g}(x) = \begin{cases} f(x), & x \in X \setminus N \\ 0, & x \in N. \end{cases}\)

Then \(\tilde{g} = f\ \overline{\mu}\text{-a.e.}\)

Moreover, if \(x \in \mathbb{Q}\), then \(\tilde{g}^{-1}((-\infty, x)) = \begin{cases} (E_x \cap (X \setminus N)) \cup N, & \text{if } 0 < x \\ E_x \cap (X \setminus N), & \text{if } x \geq 0. \end{cases}\)

Since \(E_x \cap (X \setminus N) = F_x \cap (X \setminus N) \in \mathcal{M},\) we get
that \(\tilde{g}^{-1}((-\infty, x)) \in \mathcal{M}, \forall x \in \mathbb{Q}.\)

This implies that \(g\) is \(\overline{\mu}\)-measurable.