Math 240A, make-up lecture, Monday, October 23

Terminology Let $(X, \mathcal{M})$ be a measurable space. A function $f: X \to \mathbb{R}$ is $\mathcal{M}$-measurable if it is $(\mathcal{M}, \mathcal{B}_{\mathbb{R}})$-measurable : $f^{-1}(E) \in \mathcal{M}$, $\forall E \in \mathcal{B}_{\mathbb{R}}$.

Let $\overline{\mathbb{R}} = [-\infty, +\infty]$ the extended real numbers

$\mathcal{B}_{\overline{\mathbb{R}}} = \{E \subset \overline{\mathbb{R}} \mid E \cap \mathbb{R} \in \mathcal{B}_{\mathbb{R}}\}$

$\hookrightarrow$ σ-algebra generated by $(a, +\infty]$ or $[-\infty, a)$, $a \in \mathbb{R}$.

A function $f: X \to \overline{\mathbb{R}}$ is $\mathcal{M}$-measurable if it is $(\mathcal{M}, \mathcal{B}_{\overline{\mathbb{R}}})$-measurable.

Proposition ① Let $(X, \mathcal{M})$ be a measurable space and $f_i : X \to \overline{\mathbb{R}}$, $i \geq 1$, be $\mathcal{M}$-measurable functions.

Then
$$\left.\begin{aligned} g_1(x) &= \sup_i f_i(x) \\ g_2(x) &= \inf_i f_i(x) \\ g_3(x) &= \limsup_{i\to\infty} f_i(x) \\ g_4(x) &= \liminf_{i\to\infty} f_i(x) \end{aligned}\right\} \text{ are } \mathcal{M}\text{-measurable.}$$

Moreover, if $f(x) = \lim_{i\to\infty} f_i(x)$ exists for every $x \in X$, then f is $\mathcal{M}$-measurable.

Proof Since $g_1^{-1}((a, \infty]) = \bigcup_{i\geq 1} f_i^{-1}((a,\infty]) \in \mathcal{M}$

and $g_2^{-1}((a,\infty]) = \bigcup_{n\geq 1}\left(\bigcap_{i\geq 1} f_i^{-1}\left(\left(a+\frac{1}{n}, \infty\right]\right)\right) \in \mathcal{M}$,

g_1 and g_2 are $\mathcal{M}$-measurable.

Thus, $g_3(x) = \inf_{k\geq 1}\left(\sup_{i\geq k} f_i(x)\right)$

and $g_4(x) = \sup_{k\geq 1}\left(\inf_{i\geq k} f_i(x)\right)$ are $\mathcal{M}$-measurable.

Moreover, if f exists, then $f = g_1 = g_2$ is $\mathcal{M}$-measurable $\boxtimes$

Corollary ② If $f, g : X \to \overline{\mathbb{R}}$ are measurable, then

$\max(f, g)$

$\min(f, g)$

$f^+(x) = \max(f(x), 0) \geq 0$

$f^-(x) = -\min(f(x), 0) \geq 0$ are all measurable.

Notation f^+ and f^- are called the positive and negative parts of f. Note that $f = f^+ - f^-$.

Remark If $f : X \to \mathbb{C}$ is measurable, then

$f = (\operatorname{sgn} f) \cdot |f|$ is called the polar decomposition of f.

Since $\operatorname{sgn} : \mathbb{C} \to \mathbb{C}$ is Borel measurable (exercise) $\left(\operatorname{sgn} z = \begin{cases} \frac{z}{|z|}, & z \neq 0 \\ 0, & z = 0 \end{cases} \right)$

and $|\cdot| : \mathbb{C} \to \mathbb{C}$ is continuous,

we have that $\operatorname{sgn} f = \operatorname{sgn} \circ f$ and $|f| = |\cdot| \circ f$ are measurable.

Goal $(X, \mathcal{M}, \mu)$ measure space.

Define $\int_X f \, d\mu$, for $f \geq 0$ measurable.

1) first for characteristic functions $\left(\int_X 1_E \, d\mu = \mu(E) \right)$
and their ~~linear combinations~~.

2) second, approximate any $f \geq 0$ by simple functions f_n and define
$$\int_X f \, d\mu = \lim_{n \to \infty} \int_X f_n \, d\mu$$

3) extend by linearity for complex-valued functions:
$f : X \to \mathbb{C}$ $\quad f = f_1 + i f_2 = (f_1^+ - f_1^-) + i(f_2^+ - f_2^-)$

Let $(X,\mathcal{M})$ be a measurable space

<u>Definition</u> A <u>simple</u> function $f: X \to \mathbb{C}$ is a finite linear combination of characteristic functions of sets in $\mathcal{M}$,

i.e. $$f = \sum_{i=1}^{m} c_i \chi_{F_i}, \quad \text{for some } \begin{array}{l} c_1, \dots, c_m \in \mathbb{C} \\ F_1, \dots, F_m \in \mathcal{M}. \end{array}$$

<u>Remarks</u> 1) f is a simple function $\iff$ f is measurable and $f(X) \subset \mathbb{C}$ is finite.

2) If f is a simple function and $f(X) = \{z_1, \dots, z_m\}$, then $f = \sum_{i=1}^{n} z_i \chi_{E_i}$, where $E_i = f^{-1}(\{z_i\})$, is called the <u>standard representation</u>.

<u>Theorem ③</u> Let $(X,\mathcal{M})$ be a measurable space, and $f: X \to [0,+\infty]$ be a measurable function.

Then $\exists$ an increasing sequence of simple functions $\{\varphi_n\}_{n\geq 1}$ such that

$\varphi_1 \leq \varphi_2 \leq \dots \leq \varphi_n \leq f$

① $\varphi_n \nearrow f$ pointwise, and

② $\varphi_n \to f$ uniformly on any set on which f is bounded

<u>Proof</u> Let $$\varphi_n(x) = \begin{cases} \dfrac{i-1}{2^n}, & \text{if } f(x) \in \left[\dfrac{i-1}{2^n}, \dfrac{i}{2^n}\right) \\ & \text{for some } i = 1, 2, \dots, 2^n n \\ n, & \text{if } f(x) \geq n \end{cases}$$

If $f(x) < n$, then $f(x) \in \left[\dfrac{i-1}{2^n}, \dfrac{i}{2^n}\right) = \left[\dfrac{2(i-1)}{2^{n+1}}, \dfrac{2i-1}{2^{n+1}}\right) \sqcup \left[\dfrac{2i-1}{2^{n+1}}, \dfrac{2i}{2^{n+1}}\right)$

which shows that $\varphi_n(x) \leq \varphi_{n+1}(x)$.

If $f(x) \geq n$, then $f(x) \geq \dfrac{n 2^{n+1}}{2^{n+1}} \Rightarrow \varphi_{n+1}(x) \geq n = \varphi_n(x)$.

This altogether proves that $\varphi_n \leq \varphi_{n+1}$.

If $f(x)=+\infty$, then $\varphi_n(x)=n \nearrow f(x)$.
If $f(x)\in[0,\infty)$, let $N\in\mathbb{N}$ s.t. $f(x)<N$.
Then $\forall n\geq N$, $0\leq f(x)<n \Rightarrow$

$$0\leq f(x)-\varphi_n(x)<\frac{1}{2^n},$$ again proving that $\varphi_n(x)\nearrow f(x)$.

For ②, let $E\in\mathcal{M}$ such that $\exists C>0$ satisfying
$0\leq f(x)\leq C$, $\forall x\in E$.
Then $\forall n>C$, we have $0\leq f(x)-\varphi_n(x)<\frac{1}{2^n}$, $\forall x\in E$,
proving that $\varphi_n\to f$ uniformly on E $\boxtimes$

<u>Corollary ④</u> If $f: X\to\mathbb{C}$ is measurable, then $\exists$ a sequence of simple functions $\{\Phi_n\}_{n\geq1}$ such that

① $\{|\Phi_n|\}$ is an increasing sequence
② $\Phi_n\to f$ pointwise.
③ $\Phi_n\to f$ uniformly on any set on which f is bounded.

<u>Proof</u> Let $f=g+ih$

$$=(g^+-g^-)+i(h^+-h^-),$$ where $g^\pm, h^\pm: X\to[0,\infty)$ are measurable.

Let $0\leq\psi_n^\pm\nearrow g^\pm$ and $0\leq\eta_n^\pm\nearrow h^\pm$ be the sequences of simple functions provided by Thm ③.
Then ② and ③ are clear for

$$\Phi_n:=(\psi_n^+-\psi_n^-)+i(\eta_n^+-\eta_n^-)$$

① is also easy to check since

$$|\Phi_n|=\sqrt{(|\psi_n^+|+|\psi_n^-|)^2+(|\eta_n^+|+|\eta_n^-|)^2}\nearrow\sqrt{(|g^+|+|g^-|)^2+(|h^+|+|h^-|)^2}$$

$$\| \quad |f|. \ \boxtimes$$

Proposition ⑤ (exercise)
Assume $(X, \mathcal{M}, \mu)$ is a measure space, with μ complete.
Then 1) f measurable and $g = f$ μ-a.e. $\Rightarrow$ g measurable
2) f_n measurable, $\forall n \geq 1$, and $f_n \to f$ μ-a.e.,
then f is measurable.

Proposition ⑥ Let $(X, \mathcal{M}, \mu)$ be a measure space, and $(X, \overline{\mathcal{M}}, \overline{\mu})$ be its completion.
If $f : X \to \mathbb{R}$ is $\overline{\mathcal{M}}$-measurable, then $\exists\, g : X \to \mathbb{R}$ $\mathcal{M}$-measurable such that $g = f$ $\overline{\mu}$-a.e.

Proof For $r \in \mathbb{Q}$, let $E_r = f^{-1}((-\infty, r)) \in \overline{\mathcal{M}}$.
By the construction of $\overline{\mathcal{M}}$ $\Rightarrow$ $\exists\, F_r \in \mathcal{M}$ such that
$$E_r \subset F_r \text{ and } \overline{\mu}(F_r \setminus E_r) = 0.$$
Moreover, $F_r \setminus E_r \subset N_r \in \mathcal{M}$ with $\mu(N_r) = 0$.
Let $N = \bigcup_{r \in \mathbb{Q}} N_r \in \mathcal{M}$. Then $\mu(N) = 0$.
We define $$g(x) = \begin{cases} f(x), & x \in X \setminus N \\ 0, & x \in N. \end{cases}$$
Then $g = f$ $\overline{\mu}$-a.e.
Moreover, if $r \in \mathbb{Q}$, then $$g^{-1}((-\infty, r)) = \begin{cases} (E_r \cap (X \setminus N)) \cup N, & \text{if } 0 < r \\ E_r \cap (X \setminus N), & \text{if } r \geq 0. \end{cases}$$

Since $E_r \cap (X \setminus N) = F_r \cap (X \setminus N) \in \mathcal{M}$, we get that $g^{-1}((-\infty, r)) \in \mathcal{M}$, $\forall r \in \mathbb{Q}$.

This implies that g is $\mathcal{M}$-measurable $\boxtimes$