Name: ______________________________ **PID:** ______________________

Math 140C: Final Exam
Foundations of Real Analysis

- You have 3 hours. No books and notes are allowed.
- You may quote any result stated in the textbook or in class.
- State carefully the hypothesis and conclusion of any result that you use.
- You may not use homework problems (without proof) in your solutions.

1. (10 points)

(a) (3 points) Define what it means for a function $f : \mathbb{R}^2 \to \mathbb{R}$ to be differentiable at $(0, 0)$.

(b) (7 points) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function such that the partial derivatives $\frac{\partial f}{\partial x}(x, y)$ and $\frac{\partial f}{\partial y}(x, y)$ exist and satisfy that $|\frac{\partial f}{\partial x}(x, y)| \leqslant |x|$ and $|\frac{\partial f}{\partial y}(x, y)| \leqslant |y|$, for all $(x, y) \in \mathbb{R}^2$.
Prove that f is differentiable at $(0, 0)$ and $f'(0, 0) = 0$.

2. (10 points) Consider the function $f : \mathbb{R}^3 \to \mathbb{R}$ given by $f(x, y, z) = x^5y^4 + y^3z^2 + x - 3$. Define $F : \mathbb{R}^3 \to \mathbb{R}^3$ by letting $F(x, y, z) = (f(x, y, z), y, z)$.

(a) (4 points) Prove that there exists an open set $U \subset \mathbb{R}^2$ which contains $(1, 1)$ and a differentiable function $g : U \to \mathbb{R}$ such that $g(1, 1) = 1$ and $f(g(y, z), y, z) = 0$, for all $(y, z) \in U$.

(b) (2 points) Find the partial derivatives $\dfrac{\partial g}{\partial y}(1, 1)$ and $\dfrac{\partial g}{\partial z}(1, 1)$.

(c) (4 points) Prove that $F(V)$ is an open subset of $\mathbb{R}^3$, for every open set $V \subset \mathbb{R}^3$.

3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function such that the first-order partial derivatives $\dfrac{\partial f}{\partial x}(x, y)$, $\dfrac{\partial f}{\partial y}(x, y)$, and the second-order partial derivative $\dfrac{\partial^2 f}{\partial x \partial y}(x, y)$ exist at every $(x, y) \in \mathbb{R}^2$.

(a) (5 points) Assume that $\dfrac{\partial^2 f}{\partial x \partial y}$ is continuous at $(0, 0)$. Prove that

$$\lim_{t \to 0} \frac{f(t, t^2) - f(t, 0) - f(0, t^2) + f(0, 0)}{t^3} = \frac{\partial^2 f}{\partial x \partial y}(0, 0).$$

(b) (5 points) Assume that $\dfrac{\partial^2 f}{\partial x \partial y}(x, y) = 0$, for every $(x, y) \in \mathbb{R}^2$. Prove that there exist two differentiable functions $g : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ such that $f(x, y) = g(x) + h(y)$, for every $(x, y) \in \mathbb{R}^2$.

4. (10 points) Consider the measurable space $X = \mathbb{R}$ in which the σ-ring $\mathscr{M}$ is the set of all Lebesgue measurable subsets of $\mathbb{R}$.

(a) (3 points) Assume that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function. Prove that f is measurable.

(b) (3 points) Assume that $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function. Prove that the derivative function $f' : \mathbb{R} \to \mathbb{R}$ is measurable.

(c) (4 points) Assume that $f : \mathbb{R} \to \mathbb{R}$ is a monotonically increasing function. Prove that f is measurable.

5. (10 points)

(a) (5 points) Let $A = \cup_{n=1}^{\infty}[n, n + \frac{1}{n^2}]$.
Prove that if a function $f : A \to \mathbb{R}$ satisfies $f \in \mathscr{L}^2$ on A, then $f \in \mathscr{L}$ on A.

(b) (5 points) Let $B = \cup_{n=1}^{\infty}[n, n + \frac{1}{\sqrt{n}}]$.
Give an example of a function $f : B \to \mathbb{R}$ such that $f \in \mathscr{L}^2$ on B and $f \notin \mathscr{L}$ on B.

6. (10 points) Let $f : [a, b] \to \mathbb{R}$ be a function such that $f \in \mathscr{L}$ on $[a, b]$.

Define $F : [a, b] \to \mathbb{R}$ by letting $F(x) = \displaystyle\int_a^x f(t)\, dt$.

(a) (5 points) Prove that F is continuous on $[a, b]$.

(b) (5 points) Assume that f is Riemann integrable on $[a, b]$. Prove that $F'(x)$ exists and $F'(x) = f(x)$, almost everywhere on $[a, b]$.

7. (10 points) Let $f : [0,2] \to \mathbb{R}$ be a function such that $f \in \mathscr{L}^2$ on $[0,2]$. Let $g : [0,1] \to \mathbb{R}$ be a function such that $g \in \mathscr{L}^2$ on $[0,1]$. Let $A \subset [0,1]$ be a Lebesgue measurable set such that $m(A) > 0$, where m denotes the Lebesgue measure. For $t \in \mathbb{R}$, let $A+t = \{x+t | x \in A\}$.

(a) (4 points) Prove that $\lim_{n\to\infty} \int_0^1 |f(x + \frac{1}{n}) - f(x)|^2 \, \mathrm{d}x = 0$.

(b) (3 points) Prove that $\lim_{n\to\infty} \int_0^1 f(x + \frac{1}{n}) g(x) \, \mathrm{d}x = \int_0^1 f(x) g(x) \, \mathrm{d}x$.

(c) (3 points) Prove that there exists $n \geqslant 1$ such that $(A + \frac{1}{n}) \cap A \neq \emptyset$.

8. (10 points) Let X be a measurable space, in which $\mathscr{M}$ is the σ-ring of measurable sets and μ is the measure. Let $f : X \to \mathbb{R}$ be a function such that $f \in \mathscr{L}(\mu)$. Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of sets such that $A_n \in \mathscr{M}$, for all $n \geqslant 1$.

(a) (2 points) Prove that if $A \in \mathscr{M}$, then $\displaystyle\int_A f \, \mathrm{d}\mu = \int_X f \, \mathbf{1}_A \, \mathrm{d}\mu$.

(b) (3 points) Assume that $\displaystyle\sum_{n=1}^{\infty} \mu(A_n) < +\infty$. Prove that $\displaystyle\lim_{n\to\infty} \mathbf{1}_{A_n}(x) = 0$, almost everywhere on X.

(c) (3 points) Assume that $\displaystyle\sum_{n=1}^{\infty} \mu(A_n) < +\infty$. Prove that $\displaystyle\lim_{n\to\infty} \int_{A_n} f \, \mathrm{d}\mu = 0$.

(d) (2 points) Assume that $\displaystyle\lim_{n\to\infty} \mu(A_n) = 0$. Prove that $\displaystyle\lim_{n\to\infty} \int_{A_n} f \, \mathrm{d}\mu = 0$.

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1		out of 10 points
2		out of 10 points
3		out of 10 points
4		out of 10 points
5		out of 10 points
6		out of 10 points
7		out of 10 points
8		out of 10 points
Total		out of 40 points