Name: ________________________________ **PID:** ______________________

Math 140C: Midterm 1
Foundations of Real Analysis

- You have 1 hour and 20 minutes. No books and notes are allowed.
- You may quote any result stated in the textbook or in class.
- State carefully the hypothesis and conclusion of any result that you use.
- You may not use homework problems (without proof) in your solutions.

1. (10 points) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function such that the partial derivatives $\dfrac{\partial f}{\partial x}(x, y)$ and $\dfrac{\partial f}{\partial y}(x, y)$ exist and satisfy $|\dfrac{\partial f}{\partial x}(x, y)| \leqslant 1$ and $|\dfrac{\partial f}{\partial y}(x, y)| \leqslant 1$, for every $(x, y) \in \mathbb{R}^2$.

Prove that $|f(x, y) - f(0, 0)| \leqslant |x| + |y|$, for every $(x, y) \in \mathbb{R}^2$.

2. (10 points) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = \begin{cases} \dfrac{x^3 y}{x^4 + y^2}, & \text{if } \ (x, y) \neq (0, 0) \\ 0, & \text{if } \ (x, y) = (0, 0) \end{cases}$

(a) (3 points) Prove that $|f(x, y)| \leqslant \dfrac{|x|}{2}$, for every $(x, y) \in \mathbb{R}^2$. Use this inequality to conclude that f is continuous at $(0, 0)$.

(b) (3 points) Prove that $\lim\limits_{t \to 0} \dfrac{f(th)}{t} = 0$, for every $h \in \mathbb{R}^2$.

(c) (2 points) Prove that $\lim\limits_{t \to 0} \dfrac{f(t, t^2)}{t} \neq 0$.

(d) (2 points) Prove that f is not differentiable at $(0, 0)$.

3. (10 points) Let $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ be a continuously differentiable mapping such that $\|\varphi'(x)\| < \frac{1}{2}$, for every $x \in \mathbb{R}^n$. Define $f : \mathbb{R}^n \to \mathbb{R}^n$ by $f(x) = x - \varphi(x)$, for every $x \in \mathbb{R}^n$.

(a) (3 points) Prove that there exists $x \in \mathbb{R}^n$ such that $f(x) = 0$.

(b) (3 points) Prove that f is 1-1.

(c) (2 points) Prove that $f'(x)$ is invertible, for every $x \in \mathbb{R}^n$.

(d) (2 points) Prove that $f(\mathbb{R}^n)$ is an open subset of $\mathbb{R}^n$.

4. (10 points) Consider the function $f : \mathbb{R}^3 \to \mathbb{R}$ given by $f(x, y_1, y_2) = xy_1y_2 + x^5 - 2$.

(a) (6 points) Prove that there exist an open set $U \subset \mathbb{R}^2$ which contains $(1,1)$ and a differentiable function $g : U \to \mathbb{R}$ such that $g(1,1) = 1$ and $f(g(y_1, y_2), y_1, y_2) = 0$, for all $(y_1, y_2) \in U$.

(b) (4 points) Find the partial derivatives $\dfrac{\partial g}{\partial y_1}(1,1)$ and $\dfrac{\partial g}{\partial y_2}(1,1)$.

Do not write on this page.

1		out of 10 points
2		out of 10 points
3		out of 10 points
4		out of 10 points
Total		out of 40 points