Math 142A : Practice Midterm 2

1. True/False: Circle the correct answer. No justifications are needed in this exercise.

(1) Every bounded sequence (a_n) is Cauchy. **T / F**

(2) Assume that (a_n) is a bounded sequence of real numbers which does not converge. Then $\liminf a_n < \limsup a_n$. **T / F**

(3) Every sequence (a_n) has a convergent subsequence. **T / F**

(4) Every bounded sequence (a_n) has a Cauchy subsequence. **T / F**

(5) The series $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{8^n}$ is convergent. **T / F**

2. (a) Define what it means for a sequence (a_n) of real numbers to be Cauchy.

(b) Let (a_n) be a sequence of real numbers.
Define what it means for a real number a to be a subsequential limit of (a_n).

(c) Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers.
Define what it means for the series $\sum_{n=1}^{\infty} a_n$ to be absolutely convergent.

3. (a) Let (a_n) and (b_n) be sequences of real numbers such that $|b_m - b_n| \leq |a_m - a_n|$, for every $m, n \in \mathbb{N}$. Assume that (a_n) is convergent. Prove that (b_n) is convergent.

(b) Let (a_n) be the sequence given by $a_n = \dfrac{(-1)^n n}{2n+1}$. Prove that $\limsup a_n = \frac{1}{2}$.

4. (a) Let (a_n) be an unbounded sequence of real numbers. Prove that there exists a subsequence (a_{n_k}) of (a_n) such that $\lim_{k\to\infty} |a_{n_k}| = +\infty$.

(b) Let (a_n) be a sequence of real numbers which does not converge to a real number a. Prove that there exist $\varepsilon > 0$ and a subsequence (a_{n_k}) of (a_n) such that $|a_{n_k} - a| \geq \varepsilon$, for every $k \in \mathbb{N}$.

5. (a) Given an example of a convergent but not absolutely convergent series $\sum_{n=1}^{\infty} a_n$.

(b) Prove that if $\sum_{n=1}^{\infty} a_n$ is a convergent series of positive numbers, then the series $\sum_{n=1}^{\infty} a_n^2$ converges.

(c) Let (a_n) be a sequence of real numbers. Assume that the series $\sum_{n=1}^{\infty} a_n b_n$ converges, for any bounded sequence (b_n). Prove that the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.