Math 142A : Additional practice problems

1. True/False: Circle the correct answer. No justifications are needed in this exercise.

(1) Let \( f, g : \mathbb{R} \to \mathbb{R} \) be two functions which are uniformly continuous on \( \mathbb{R} \).
Then \( fg \) is uniformly continuous on \( \mathbb{R} \). \( \text{T / F} \)

(2) There is a continuous function \( f : (0, 1) \to \mathbb{R} \) such that \( f((0,1)) = (0, \infty) \). \( \text{T / F} \)

(3) There is a continuous function \( f : [0, 1] \to \mathbb{R} \) such that \( f([0,1]) = [0, \infty) \). \( \text{T / F} \)

2. (a) Let \( S \) be a subset of \( \mathbb{R} \) and \( f : S \to \mathbb{R} \) be a function.
Define what it means for \( f \) to be continuous at \( x_0 \in S \).

(b) Let \( f : \mathbb{R} \to \mathbb{R} \) be a function and \( L \) be a real number.
Define what it means to have that \( \lim_{x \to a} f(x) = L \), for some \( a \in \mathbb{R} \).

3. Consider the function \( f : (0, \infty) \to \mathbb{R} \) given by \( f(x) = \frac{1}{\sqrt{x}} \).

(a) Prove that \( f \) is continuous at \( x_0 = 1 \) by verifying the \( \varepsilon \)-\( \delta \) property.

(b) Prove that \( f \) is not uniformly continuous on \( (0, \infty) \).

4. (a) Prove that there exists \( x \in (0, 2) \) such that \( x^5 = 4x + 3 \).

(b) Let \( f : [0, 1] \to \mathbb{R} \) be a function which is continuous on \([0, 1]\) and satisfies \( f(0) = f(1) = \frac{1}{2} \).
Prove that there exists \( x \in (0, 1) \) such that \( f(x) = 1 - x \).

5. Let \( M > 0 \) and \( f : \mathbb{R} \to \mathbb{R} \) be a function such that \( |f(x) - f(y)| \leq M|x - y| \), for all \( x, y \in \mathbb{R} \).
Prove that \( f \) is uniformly continuous on \( \mathbb{R} \).