Math 142B : Additional practice problems

1. True/False: Circle the correct answer. No justifications are needed in this exercise.
   (1) If $f : [0, 1] \to \mathbb{R}$ is integrable on $[0, 1]$, then $f$ is continuous on $[0, 1]$.  
      $\quad \text{T} / \text{F}$
   (2) Let $(f_n)$ be a sequence of integrable functions on $[0, 1]$. Assume that $\lim_{n \to \infty} f_n(x) = 0$, for every $x \in [0, 1]$. Then $\lim_{n \to \infty} \int_0^1 f_n = 0$. 
      $\quad \text{T} / \text{F}$
   (3) If $f$ is continuous function on $\mathbb{R}$, then there exists a function $F$ which is differentiable on $\mathbb{R}$ such that $f = F'$. 
      $\quad \text{T} / \text{F}$

2. (a) State the Intermediate Value Theorem for Integrals.
   (b) State what it means for a function $f : [a, b] \to \mathbb{R}$ to be piecewise continuous.

3. Let $f : [a, b] \to \mathbb{R}$ be an increasing function.
   Prove that $U(f, P) - L(f, P) \leq \text{mesh}(P) \cdot [f(b) - f(a)]$, for every partition $P$ of $[a, b]$.

4. Let $f$ and $g$ be continuous functions on $[0, 1]$. Assume that $g(x) > 0$, for every $x \in [0, 1]$.
   Prove that there exists $x \in (0, 1)$ such that $\frac{f(x)}{g(x)} = \frac{\int_0^1 f}{\int_0^1 g}$.

5. Let $g : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $g(\mathbb{R}) = \mathbb{R}$ and $g'(x) > 0$, for all $x \in \mathbb{R}$. Let $G : \mathbb{R} \to \mathbb{R}$ be given by $G(x) = \int_0^{g(x)} g^{-1}(t) \, dt$, for every $x \in \mathbb{R}$.
   Prove that $G$ is differentiable on $\mathbb{R}$ and $G'(x) = xg'(x)$, for every $x \in \mathbb{R}$. 
