1. True/False: Circle the correct answer. No justifications are needed in this exercise.

(1) Let \( f : \mathbb{R} \to \mathbb{R} \) be a function which is differentiable at 0. Then \( \lim_{x \to 0} \frac{f(x^3) - f(0)}{x^2} = 0 \).  \( \text{T/F} \)

(2) Let \( f : \mathbb{R} \to \mathbb{R} \) be a function which is differentiable on \( \mathbb{R} \) and satisfies \( f'(x) \geq 0 \), for every \( x \in \mathbb{R} \). Then \( f \) is strictly increasing on \( \mathbb{R} \).  \( \text{T/F} \)

(3) Let \( f : (0, 1) \cup (2, 3) \to \mathbb{R} \) be a differentiable function such that \( f'(x) = 0 \), for every \( x \in (0, 1) \cup (2, 3) \). Then \( f \) is a constant function on \( (0, 1) \cup (2, 3) \).  \( \text{T/F} \)

(4) Let \( f : \mathbb{R} \to \mathbb{R} \) be a function which has derivatives of all orders at every \( x \in \mathbb{R} \). Assume that the Taylor series for \( f \) about 0 is identically zero. Then \( f \) is identically zero.  \( \text{T/F} \)

(5) Let \( f : [0, 1] \to \mathbb{R} \) be a bounded function which is integrable on \([0, 1]\). Then we have \( \int_0^1 f = \lim_{n \to \infty} \sum_{k=1}^{\infty} f\left(\frac{k}{n}\right) \).  \( \text{T/F} \)

2. (a) State the chain rule theorem.
(b) Let \( f \) be a real-valued function defined on an interval \( I \). Define what it means for \( f \) to be decreasing on \( I \).
(c) Let \( f : [a, b] \to \mathbb{R} \) be a bounded function. Let \( P = \{a = t_0 < t_1 < \ldots < t_n = b\} \) be a partition of \([a, b]\). Give the definition of a Riemann sum of \( f \) associated with \( P \).

3. (a) Let \( f : \mathbb{R} \to \mathbb{R} \) be defined by \( f(x) = \begin{cases} x^2, & \text{if } x \text{ is rational} \\ x^3, & \text{if } x \text{ is irrational} \end{cases} \). Prove that \( f \) is differentiable at 0 and \( f'(0) = 0 \).
(b) Let \( f : \mathbb{R} \to \mathbb{R} \) be a differentiable function such that \( f'(x) < 1 \), for every \( x \in \mathbb{R} \). Prove that the equation \( f(x) = x \) has at most one solution \( x \in \mathbb{R} \).

4. (a) Show that \( \lim_{x \to \infty} \sqrt{x} \sin \left(\frac{1}{x}\right) = 0 \).
(b) Use Taylor’s theorem to prove that \( 1 + \frac{3}{2}x < (1 + x)^{\frac{3}{2}} < 1 + \frac{3}{2}x + \frac{3}{8}x^2 \), for every \( x > 0 \).
(c) Let \( f : \mathbb{R} \to \mathbb{R} \) be a function which has derivatives of all orders such that \( |f^{(n)}(x)| \leq n! \), for any \( x \in (-1, 1) \) and \( n \in \mathbb{N} \). Prove that \( f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k \), for every \( x \in (-1, 1) \).

5. Let \( f : [0, 1] \to \mathbb{R} \) be a function defined by \( f(x) = \begin{cases} 2x, & \text{if } x \in [0, 1] \text{ is rational} \\ x + 1, & \text{if } x \in [0, 1] \text{ is irrational} \end{cases} \). Let \( n \geq 1 \) and consider the partition \( P_n = \{0 = t_0 < t_1 < \ldots < t_n = 1\} \), where \( t_k = \frac{k}{n} \).
(a) Calculate \( L(f, P_n) \).
(b) Calculate \( U(f, P_n) \).
(c) Prove that \( f \) is not integrable on \([0, 1]\).