HW1

Q5 on p.130

$g(z)$ has all removable singularity at $z_1, \ldots, z_k \in B\left(0, \frac{1}{3}R\right)$. Moreover, by maximum modulus theorem, it follows that

$$\max_{|z|\leq R} g(z) \leq \max_{|z|=R} g(z) = \max_{|z|=R} \left| \frac{f(z)}{\prod_{k=1}^{n}\left(1-\frac{z}{z_k}\right)} \right|$$
$$\leq \frac{M}{\prod_{k=1}^{n} \min_{|z|=R}\left|1-\frac{z}{z_k}\right|}$$
$$\leq \frac{M}{\prod_{k=1}^{n} \min_{|z|=R}\left|\frac{z}{z_k}\right|-1}$$
$$\leq \frac{M}{\prod_{k=1}^{n} \frac{R}{\frac{1}{3}R}-1} = M2^{-n}.$$

Therefore, we have $a = f(0) = g(0) \leq M2^{-n}$. Taking log on both sides, we obtain

$$n \leq \frac{1}{\log 2}\log\left(\frac{M}{a}\right).$$

Q3 on p.133

(a) Suppose there exist $z_0 \in D$ such that f $(z) = 0$, then by using a minmium modulus theorem on the function e^f and the fact that $\mathrm{Re}(f) \geq 0$ on D it follows that

$$\left|e^{f(z_0)}\right| \leq \left|e^{f(z)}\right| = e^{\mathrm{Re}\, f(z)} \text{ for } z \in B_r(z_0)$$

where $B_r(z_0)$ is an open ball with center z_0 and radius r such that $B_r(z_0) \subseteq D$. Since e^f attains a minimum on D, e^f is a constnat on D which in turn f is a constant on D. It has a contradiction. Hence, $\mathrm{Re}(f) > 0$ on D.

(b) Let $F(w) = \frac{w-1}{w+1}$ maps right plane on $\mathbb{C}$ to D. Then, we have $F \circ f$ maps D to itself with $F \circ f(0) = 0$. By using the Schwartz Lemma, we have

$$\left|\frac{f-1}{f+1}\right| \leq |z|. \tag{0.1}$$

equ.1.1

Therefore, it implies

$$\frac{|f|-1}{|f|+1} \leq |z|,$$

and $|f(z)| \leq \frac{1+|z|}{1-|z|}$ follows.

(c) Using Eq. (0.1) equ.1.1 and the triangle inequality that $1-|f| \leq |f-1|$, the result follows.

Q6 on p.133

Suppose f never vanish in D, f is analytic on D, f is continuous on $\overline{D}$ and f maps ∂D to itself, by using maximum and minimum modulus principle, we can conclude that for $z \in D$, we have

$$1 = \min_{w \in \partial D} |f(w)| \leq |f(z)| \leq \max_{w \in \partial D} |f(w)| = 1.$$

Hence, $f(z)$ is constantly equal to $f(0)$ for $z \in D$.

In general, suppose f vanishes at $z_1, \ldots, z_k \in D$ with multiplicity $n_1, \ldots, n_k$. We let

$$F(z) = f(z) / \prod_{j=1}^{k} \left(\frac{z - z_j}{1 - \overline{z_j} z} \right)^{n_j}.$$

We can check that $F(z)$ maps ∂D to ∂D and $F(z)$never vanishes. Therefore, we have

$$F(z) = F(0) = \frac{f(0)}{(-1)^{n_1 + \cdots + n_k} \prod_{j=1}^{k} z_j^{n_j}}$$

and hence,

$$f(z) = F(0) \prod_{j=1}^{k} \left(\frac{z - z_k}{1 - \overline{z_k} z} \right).$$

Q4 on p.138

Let $g(z) = e^z$. Then $g : \{z \in \mathbb{C} \ : \ \log R_1 < \operatorname{Re}(z) < \log R_2\} \to \operatorname{ann}(0; R_1, R_2)$ is a bijective analytic map. Then, we can apply Theorem 3.7 (see the textbook p.135) on $f \circ g$. We have $f \circ g$ to be a log-convex. In other words, for any $\log R_1 < w_1 < w < w_2 \leq \log R_2$ we have

$$\log \hat{M}(w) \leq \frac{w_2 - w}{w_2 - w_1} \log \hat{M}(w_1) + \frac{w - w_1}{w_2 - w_1} \log \hat{M}(w_2)$$

where $\hat{M}(w) := \max\{|f \circ g(z)| \, z = w\}$. If we define $M(r) := \max\{|f(z)| \, z = r\}$, then Hadamard's Three Circles Theorem is followed by observing $\hat{M}(\log r) = M(r)$, $w_2 = \log r_2$, $w_1 = \log r_1$ and $w = \log r$.

Q6 on p.138

We fixed $r < R$. It is checked that $F_r(z) = \frac{1}{2\pi} \int f\left(ze^{i\theta}\right) \varphi_r(\theta) \, d\theta$ is analytic on $B(0, R)$ where $\varphi_r(\theta) = e^{-iarg\left(f\left(re^{i\theta}\right)\right)}$ is a continous function. Hence,

$$\max_{z \in \partial B(0,r)} |F_r(z)| \leq \int_0^{2\pi} \left| f\left(re^{i\theta}\right) \right| d\theta = F_r(r) \leq \max_{z \in \partial B(0,r)} |F_r(z)|$$

and

$$F_r(r) = I(r).$$

Therefore, we have

$$\max_{z \in \partial B(0,r)} |F_r(z)| = I(r).$$

Also, supose $\hat{r} \neq r$, then $\max_{z \in \partial B(0,\hat{r})} |F_r(z)| \leq \int \int_0^{2\pi} \left| f\left(\hat{r} e^{i\theta}\right) \right| d\theta \leq I(\hat{r})$. Suppose $r_1 < r < r_2$. By using Hadamard's three circle theorem it follows

$$\begin{aligned} \log I(r) &\leq \frac{\log r_2 - \log r}{\log r_2 - \log r_1} \log \max_{z \in \partial B(0,r_1)} |F_r(z)| + \frac{\log r - \log r_1}{\log r_2 - \log r_1} \log \max_{z \in \partial B(0,r_2)} |F_r(z)| \\ &\leq \frac{\log r_2 - \log r}{\log r_2 - \log r_1} \log I(r_1) + \frac{\log r - \log r_1}{\log r_2 - \log r_1} \log I(r_2) \end{aligned}$$

Moreover, by using a maximum modulus principle, it it clear that

$$I(r_1) = \max_{z \in B(0,r_1)} |F_{r_1}(z)| \leq \max_{z \in \partial B(0,r_2)} |F_{r_1}(z)| \leq I(r_2)$$

whenever $r_2 \geq r_1$. Furthermore, suppose $I(r_2) = I(r_1)$ for $r_1 < r_2$. Then $F_{r_2}(z)$ is deduced to be constant on $B(0,R)$ by using maximum modulus principle. In particularly, we have for $r_1 < r < r_2$,

$$\begin{aligned} I(r) = F_r(0) &\leq \frac{1}{2\pi}\int_0^{2\pi} |f(0)|\, d\theta \\ &\leq |f(0)| = \frac{1}{2\pi}\left|\int_0^{2\pi} f\left(re^{i\theta}\right) d\theta\right| \\ &\leq \frac{1}{2\pi}\int_0^{2\pi} \left|f\left(re^{i\theta}\right)\right| d\theta = I(r). \end{aligned}$$

Therefore, we have

$$\frac{1}{2\pi}\left|\int_0^{2\pi} f\left(re^{i\theta}\right) d\theta\right| = \frac{1}{2\pi}\int_0^{2\pi} \left|f\left(re^{i\theta}\right)\right| d\theta. \tag{0.2}$$

equ.1.2

Let $e^{i\phi}$ be the unit norm complex number such that

$$\frac{1}{2\pi}\left|\int_0^{2\pi} f\left(re^{i\theta}\right) d\theta\right| = e^{i\phi}\frac{1}{2\pi}\int_0^{2\pi} f\left(re^{i\theta}\right) d\theta.$$

Therefore, using Eq. (0.2), we have

$$\int_0^{2\pi} \mathrm{Re}\left(f\left(re^{i\theta}\right)e^{i\phi}\right) - \left|f\left(re^{i\theta}\right)\right| d\theta = 0.$$

and it implies $f\left(re^{i\theta}\right)e^{i\phi} = \left|f\left(re^{i\theta}\right)\right|$ because $\mathrm{Re}\left(f\left(re^{i\theta}\right)e^{i\phi}\right) \leq \left|f\left(re^{i\theta}\right)\right|$. Therefore, $\mathrm{Im}\left(f(z)e^{i\phi}\right) = 0$ for $r_1 < |z| = r < r_2$ and it contradicts with open mapping theorem as $\left\{f(z)e^{i\phi} \;:\; r_1 < |z| = r < r_2\right\}$ is an open set.

Remark 0.1. More elementary method is to use Cauchy-Riemann Equation instead of open mapping theorem to argue f is a constant in this case.

Q5 on p.141

[This is Professor Adrian Ioana's idea] First we let $g_n(z) = (z - z_0) f^n(z)$ for $n \in \mathbb{N}$. Then $g_n(z)$ is analytic on G. Let $C = \max_{z\in G}|z - z_0| < \infty$ as G is a bounded set. Then, we have

$$\max_{\partial G}|g_n(z)| = \max_{\partial G}|z - z_0|\,|f^n(z)| \leq CM^n.$$

Therefore, by using a maximum modulus principle 1.2, we have

$$|f(z)| \leq \frac{C^{\frac{1}{n}}M}{|z - z_0|^{\frac{1}{n}}}$$

for $z \in G$. Taking $n \to \infty$, we have $|f(z)| \leq M$.