HW2

Exercise. Q6 on p.150

First, we claim that $\{f_n\}$ is normal. Since $f(z) = \lim f_n(z)$ for $z \in G$. It follows that for each $z \in G$, there exists M_z such that $\sup_n |f_n(z)| < M_z$. Therefore, the closure of $\{f_n(z) \ : \ n \in \mathbb{N}\}$ is compact in G. Then we need the following lemma to show $\{f_n\}$ is equicontinuous on G.

lem.1.2 **Lemma 0.1.** *For any $\epsilon > 0$ and $z \in G$, there exists $\eta_z > 0$ and $N_z > 0$ such that $|f_n(w) - f(w)| < \epsilon$ for $w \in B_{\eta_z}(z) \subseteq G$ and $n \geq N_z$.*

Proof. There exists $N = N_z > 0$ such that $|f_n(z) - f(z)| < \epsilon$ for $n \geq N$. Since f and f_N are continuous at $z \in G$, there exists $\eta = \eta_z > 0$ such that $|f(z) - f(w)| < \epsilon$ and $|f_N(z) - f_N(w)| < \epsilon$ for $w \in B_\eta(z) \subseteq G$. It follows that for $n \geq N$ and $w \in B_\eta(z)$ we have

$$\begin{aligned} |f_n(w) - f(w)| &\leq |f_N(w) - f(w)| \\ &\leq |f_N(w) - f_N(z)| + |f(w) - f(z)| + |f_N(z) - f(z)| \\ &\leq 3\epsilon. \end{aligned}$$

The first inequality is because of the monotonically increasing condition of $\{f_n\}$. The lemma follows.

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For any $\epsilon > 0$ and $z \in G$, let η_z and N_z be the same symbol in Lemma 0.1 (lem.1.2). We take a $\eta_z > \delta > 0$ such that $|f(z) - f(w)| < \epsilon$ for $|z - w| < \delta$. Using Lemma 0.1 (lem.1.2), it shows that for all $n \geq N_z$

$$\begin{aligned} |f_n(w) - f_n(z)| &\leq |f_n(w) - f(w)| + |f(w) - f(z)| + |f_n(z) - f(z)| \\ &\leq 3\epsilon. \end{aligned}$$

Hence, $\{f_n\}$ is equicontinuous on G. By Arzela-Ascoli Theorem, $\{f_n\}$ is normal, i.e. f_{n_k} converges to f locally uniformly.

Next, we claim f_n converge to f locally uniformly. For any $\epsilon > 0$ and any compact set $S \subseteq G$, there exists $K > 0$ such that

$$|f_{n_k}(w) - f(w)| < \epsilon \text{ for } w \in S \text{ and } k \geq K.$$

Since $f_{n_K}(w) \leq f_m(w) \leq f(w)$ for all $m \geq n_K$, it follows that

$$|f_m(w) - f(w)| < \epsilon \text{ for } w \in S \text{ and } m \geq n_K.$$

We have f_n converges to f locally uniformly.

Remark 0.2. Another method to show $f_n \to f$: We let $g_n = f - f_n$. For any $\epsilon > 0$, $z \in K$ which is compact in G. There exists N_z such that $|g_n| \leq |g_{N_z}| = g_{N_Z} < \epsilon$. We denote $O_z = \{w \in G \ : \ g_{N_z} < \epsilon\}$. It is clear that O_z is non-empty open set and $K \subseteq \cup_{z \in K} O_z$. Since K is compact, there is a finite cover $O_{z_1}, \ldots, O_{z_k}$ to cover K. As a result, take $N = \min\{N_{z_1}, \ldots, N_{z_k}\}$ and we have $|f(z) - f_n(z)| = g_n(z) < \epsilon$ for all $z \in K$ and $n \geq N$.

Exercise. Q7 on p.150

We need one lemma before we do this question.

lem.1.1 **Lemma 0.3.** *Suppose $\{x_n\}$ is a sequence on a metric space (X, d). Then, x_n converges to x if and only if every subsequence of x_n has a furhter convergent subsequence which converges to the same limit x.*

Proof.

" $\Rightarrow$ " direction is a straight forward argument. Thus, we just show the other direction. Suppose the conclusion is false, i.e. $x_n \nrightarrow x$. Then there exist ϵ_0 and x_{n_k} such that $d(x_{n_k}, x) \geq \epsilon_0$ for all $k \in \mathbb{N}$. Therefore, no further convergent subsequence of x_{n_k} can converge to x. Contradicts with the given condition.

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Suppose $f(z) = \lim f_n(z)$ for $z \in G$. Then, for each $z \in G$, there exists M_z such that $\sup_n |f_n(z)| < M_z$. Moreover, $\{f_n\}$ is equicontinuous on G. Then, by Arzela-Ascoli Theorem, $\{f_n\}$ is normal in G. Then for any subsequence of $\{f_n\}$, there is a further convergent subsequence converging to f because of the assumption that $f(z) = \lim f_n(z)$ for $z \in G$. Hence, by using Lemma lem.1.1 0.2, $f_n \to f$.

Exercise. Q4 on p.154

$\{f_n\}_{n\in\mathbb{N}}$ is locally bounded which means $\{f_n\}_{n\in\mathbb{N}}$ is normal in $H(G)$. Then for any subsequence of $\{f_n\}$, the subsequence has a further convergent subsequence denoted by $\left\{\hat{f}_\ell\right\} \subseteq \{f_n\}$. Let $\hat{f}_\ell \to g$. By the assumption that $\lim f_n(z) = f(z)$ for $z \in A$ and A has a limit point. Therefore, $A' = \{z \in G : f(z) = g(z)\}$ has a limit point which follows that $g \equiv f$ on G. Then by using Lemma lem.1.1 0.2, we have $f_n \to f$.

Exercise. Q8 on p.154

Suppose there is a sequence of M_n of positive constant such that $|a_n| \leq M_n$ and $\limsup |M_n|^{1/n} \leq 1$. Then for any open ball with center 0 and radius $\epsilon < 1$, $B_\epsilon(0)$, we have for all $z \in B_\epsilon(0)$,

$$|f_n(z)| \leq \sum |a_n| |z|^n \leq \sum M_n \epsilon^n < \infty.$$

The last strict inequality because of $\limsup |M_n \epsilon^n|^{1/n} < 1$. It follows that $\mathcal{F}$ is locally bounded and hence normal by Montel Theorem.

Suppose $\mathcal{F}$ is normal in $H(D)$. By Montel's Theorem and Lemma 2.8, for any $0 < \epsilon < 1$, there exists $C_\epsilon > 0$ such that

$$|f(z)| \leq C_\epsilon \text{ for all } f \in \mathcal{F} \text{ and } z \in \partial B_\epsilon(0)$$

where $B_\epsilon(0)$ be a ball with center 0 and radius ϵ. Then, by using Cauchy's formula, we have

$$a_n = \frac{f^{(n)}(0)}{n!} = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z^{n+1}} dz \qquad (0.1)$$ equ.1.1

where γ is an anti-clockwisely oriented closed curve on $\partial B_\epsilon(0)$. Therefore, using Eq. equ.1.1 (0.1), $|a_n|$ is bounded by the following

$$|a_n| \leq \frac{C_\epsilon}{\epsilon^n} \text{ for all } 1 > \epsilon > 0.$$

Then we take $M_n := \inf_{0<\epsilon<1} \frac{C_\epsilon}{\epsilon^n}$. And therefore, fixed $0 < \epsilon_0 < 1$, it follows that

$$\begin{aligned}\limsup_n |M_n|^{1/n} &\leq \limsup_n \left(\inf_{0<\epsilon<1} \frac{|C_\epsilon|^{1/n}}{\epsilon} \right) \\ &\leq \limsup_n \frac{|C_{\epsilon_0}|^{1/n}}{\epsilon_o} \leq \frac{1}{\epsilon_0}\end{aligned}$$

because $|C_\epsilon|^{1/n} \to 1$ as $n \to \infty$. Therefore, $\limsup_n |M_n|^{1/n} \leq 1$ follows immediately if we take $\epsilon_0 \to 1^-$.

Exercise. Q5 on p.163,

Let D be a unit disk with center zero. By Riemann Mapping Theorem, there exists the unique biholomorphic map g from G to D such that $g(a) = 0$ and $g'(a) > 0$. Then we let $F(z) = g \circ f \circ g^{-1}(z)$ is a map from D to D such that $F(0) = 0$. By Schwarz's Lemma on p.130, we can conclude that

$$\begin{aligned}1 \geq \left|F'(0)\right| &= \left|\left(g \circ f \circ g^{-1}\right)'(z)\right| \\ &= \left|g'(a) f'(a) \left(g^{-1}\right)'(0)\right| = \left|f'(a)\right|.\end{aligned}$$

The second equality is because $f(a) = a$.