HW3

Exercise. Q3. on p.201

Lemma 0.1. *Suppose O is an open set in $\mathbb{C}_\infty$ and $Q \subseteq O$. Let D and $D^{'}$ be the connected components of O and Q respectively. If $D \cap D^{'} \neq \emptyset$, then $D^{'} \subseteq D$.*

Proof. Suppose $D^{'} \subsetneq D$. There exists $z \in D^{'} \cap \partial D \subseteq Q \subseteq O$. As O is open, there exists $\epsilon > 0$ such that a connected open ball $B_\epsilon(z) \subseteq O$. It follows that $z \in D \cap B_\epsilon(z)$ and hence, $D \cup B_\epsilon(z)$ is connected in O. Since D is open which mean $z \notin D$. Therefore, $D \cup B_\epsilon(z) \supsetneq D$. Contradiction. ■

Lemma 0.2. *Let D be the connected component of $\mathbb{C}_\infty - K$. Then $\partial D \subseteq \partial K$.*

Proof. Suppose $\partial D \nsubseteq \partial K$. Then there exists $z \in \partial D \cap K^c$. Moreover, there exists $\epsilon > 0$ such that $B_\epsilon(z) \subseteq K^c$ because K^c is open. Since $B_\epsilon(z) \cap D \neq \emptyset$ and D is a connected component in K^c. We have $B_\epsilon \subseteq D$ which mean $D \cap \partial D \neq \emptyset$. Contradiction as D is open (Note D is a connected component of a open set K^c.)
■

Proof. of Q3.

Let D be a connected component of $\mathbb{C}_\infty - K$.

Case 1. D is unbounded. Let $D^{'}$ be the unbounded component of $\mathbb{C}_\infty - G \subseteq \mathbb{C}_\infty - K$. It follows that $\infty \in D \cap D^{'}$. Therefore, by using Lemma 0.1, we have $D^{'} \subseteq D$.

Case 2. D is bounded. By the assumption that $\overline{D} \cap \partial G \neq \emptyset$, we let $z \in \overline{D} \cap \partial G$. If $z \in \partial D$, then by Lemma 0.2, we have $\partial K \cap \partial G \neq \emptyset$ which contradicts the fact that a compact $K \subseteq G$ where G is open. Therefore, $z \in D \cap \partial G$. Since D is an open connected component of K^c, there exists $\epsilon > 0$ such that $B_\epsilon(z) \subseteq D$ and hence $B_\epsilon(z) \subseteq D$. Using the fact that $B_\epsilon(z) \cap G \neq \emptyset$, we have $D \cap G \neq \emptyset$ and from Lemma 0.1, D contains a connected component of G. ■

Exercise. Q4. on p.201

From (b) to (a), by using Q3 on p.201, $\mathbb{C}_\infty - G$ meets every components of $\mathbb{C}_\infty - K$. For any f which is analytic on a neighborhood of K and $\epsilon > 0$, by Runge's Theorem, there exists rational function $g(z)$ whose poles are in $\mathbb{C}_\infty - G$ such that

$$\sup_{z \in K} |f(z) - g(z)| < \epsilon.$$

Then (a) follows because $g \in H(G)$.

From (c) to (b), suppose b is false. There exists a connected component D of G such that $\overline{D} \cap \partial G = \emptyset$. Then $\partial D \subseteq \partial K$ and $\partial D \neq \emptyset$[otherwise, D is both open and closed in $\mathbb{C}$]. For any $f \in H(G)$ and $z \in D$, we have

$$|f(z)| \le \sup_{w \in \overline{D}} |f(w)| = \sup_{w \in \partial D} |f(w)| \le \sup_{w \in \partial K} |f(w)| \le \sup_{w \in K} |f(w)|$$

where the equality is by maximum modulus principle [Thm 1.2]. Contradicting with the condition in c.

From (a) to (c), let $z_0 \in G - K$. Let $f(z) = \frac{1}{z - z_0.}$ Then by using a, for any $n \in \mathbb{N}$, there exists $f_n \in H(G)$ such that $\sup_{z \in K} |f(z) - f_n(z)| \leq \frac{1}{n}$. Hence, for all $z \in K$, it follows

$$|1 - (z - z_0) f_n(z)| \leq \frac{|z - z_0|}{n} \leq \frac{d(z_0, \partial K)}{n}.$$

Take N large enough such that $\frac{d(z_0, \partial K)}{N} \leq \frac{1}{2}$. Let $g(z) = 1 - (z - z_0) f_N(z) \in H(G)$. Then, $|g(z_0)| = 1$ while $\sup_{z \in K} |g(z)| \leq \frac{1}{2}$.

Exercise. Q1 on p.213

(a) G is path connected. Indeed for any two points w_1, $w_2 \in G$, we have two straight lines path $[a, w_1]$ and $[a, w_2] \subseteq G$. Then, $[w_1, a] \cup [a, w_2]$ is a path connecting w_1 and w_2 lying in G. [It is Alice Chan's idea] To G be open, for any $w \in G$, let $\epsilon = d([a, w], \gamma)$. For any $\hat{w} \in B_{\epsilon/2}(w)$, for any $u \in [a, \hat{w}]$, it follows that

$$d(u, [a, w]) \leq d(\hat{w}, w) \leq \epsilon/2.$$

Therefore, $u \notin \gamma$ and $\hat{w} \in G$. G being open follows immediately.

It suffices to show that $\partial G = \gamma$ in the second part. It is clear that $\gamma \subseteq \partial G$ because for any $z \in \gamma$, then we can pick $\{z_n\} \subseteq [a, z]$ such that $z_n \to z$ and $z_n \in G$. For the other inclusion, for $r \in \partial G$, $[a, r] \cap \gamma \neq \emptyset$. Let $\ell : \mathbb{C} \to \mathbb{C}$ such that $\ell(z) = [a, z, \infty) \cap \gamma$ where $[a, z, \infty)$ is an unbounded line which passing from a through z. Hence, $\ell(r) \in [a, r]$. Suppose $\ell(r) \neq r$, i.e. $r \notin \gamma$. By the continuity of ℓ, there exists an open ball B_r of r such that $\ell(\hat{r}) \in [a, \hat{r}]$ and $\ell(\hat{r}) \neq \hat{r}$ for $\hat{r} \in B_r$. It follows that $r \notin \partial G$ because there does not exists $w_n \in G$ such that $w_n \to r$. Contradiction.

(b) W.L.O.G., we assume $\gamma : [0, 1] \to \mathbb{C}$. For any $\epsilon > 0$, there exists $\epsilon > \delta > 0$ such that

$$|f(z) - f(w)| < \epsilon \text{ if } z, w \in G \text{ and } |z - w| \leq \delta + a\delta.$$

We denote a simple closed rectifiable curve $\hat{\gamma}[0, 1]$ lying in G as

$$\hat{\gamma}(t) = a + (1 - \delta)(\gamma(t) - a). \tag{0.1}$$

Then, $|f(\gamma(t) - \hat{\gamma}(t))| < \epsilon$ and $\int_{\hat{\gamma}} f = 0$. Let $0 < t_1 < \cdots < t_m < 1$ be a partition of $[0, 1]$, $\gamma_t := \gamma(t)$ and $\hat{\gamma}_t := \hat{\gamma}(t)$ such that

$$I = \left| \int_\gamma f - \sum_{k=1}^m f(\gamma_k)[\gamma_k - \gamma_{k-1}] \right| < \epsilon$$

and

$$II = \left| \int_{\hat{\gamma}} f - \sum_{k=1}^m f(\hat{\gamma}_k)[\hat{\gamma}_k - \hat{\gamma}_{k-1}] \right| < \epsilon.$$

Then,

$$\begin{aligned}\left|\int_{\gamma} f\right| &= \left|\int_{\gamma} f-\int_{\hat{\gamma}} f\right| \\ &\leq I+II+\left|\sum_{k=1}^{m} f\left(\gamma_k\right)\left[\gamma_k-\gamma_{k-1}\right]-f\left(\hat{\gamma}_k\right)\left[\hat{\gamma}_k-\hat{\gamma}_{k-1}\right]\right| \\ &\leq 2\epsilon+\sum_{k=1}^{m}\left|f\left(\gamma_k\right)-f\left(\hat{\gamma}_k\right)\right|\left|\gamma_k-\gamma_{k-1}\right|+ \\ &\sum_{k=1}^{m}\left|f\left(\hat{\gamma}_k\right)\right|\left|\left(\gamma_k-\gamma_{k-1}\right)-\left(\hat{\gamma}_k-\hat{\gamma}_{k-1}\right)\right| .\end{aligned}$$

Let $V(\gamma)$ be the total variation of γ. Then, it can be seen that

$$\sum_{k=1}^{m}\left|f\left(\gamma_k\right)-f\left(\hat{\gamma}_k\right)\right|\left|\gamma_k-\gamma_{k-1}\right| \leq \epsilon V(\gamma) .$$

Let $M=\max_{t\in[0,1]}|\gamma(t)|$. By using Eq. (0.1), it follows that

$$\sum_{k=1}^{m}\left|f\left(\hat{\gamma}_k\right)\right|\left|\left(\gamma_k-\gamma_{k-1}\right)-\left(\hat{\gamma}_k-\hat{\gamma}_{k-1}\right)\right| \leq M\sum_{k=1}^{m}\delta\left|\gamma_k-\gamma_{k-1}\right| \leq M\epsilon V(\gamma) .$$

Therefore,

$$\left|\int_{\gamma} f\right| \leq 2\epsilon+\epsilon V(\gamma)+M\epsilon V(\gamma)$$

and $\left|\int_{\gamma} f\right|=0$ is followed.

(c) $\overline{G}$ is bounded because for any $z,w\in\overline{G}$.

$$|z-w|<|z-a|+|w-a|<2\operatorname{dist}(a,\gamma)<\infty.$$

Therefore, if $z\notin\overline{G}$, then z lies in the unbounded component of $\mathbb{C}-\gamma$ and hence $n(\gamma,z)=0$ by Theorem 4.4 on p.82. For $z\in G$ where G is a region and $a\in G$. it follows that $n(\gamma,z)=n(\gamma,a)$ by Theorem 4.4 on p.82. Let $\epsilon>0$ such that $a+\epsilon e^{2\pi it}\in G$ for $t\in[0,1]$. Then, there exists a strictly monotone function $\sigma(t)$from $[0,1]$ to $[0,1]$ such that $\gamma(t)=\ell\left(a+\epsilon e^{2\pi i\sigma(t)}\right)$ for $t\in[0,1]$. Let

$$\hat{\gamma}(t)=a+\epsilon e^{2\pi i\sigma(t)}.$$

Let

$$F(t,\tau)=\tau\gamma(t)+(1-\tau)\hat{\gamma}(t).$$

It can be verified that γ is homotopic to $\hat{\gamma}$. Hence $n(\gamma,a)=n(\hat{\gamma},a)=\pm1$.

Exercise. Q7. on p.217

First, we note that $0\in T$ because $h(f_0(z))=z$ for $z\in D_0$ by the assumption. Moreover, suppose $t\in T$, then there exists an open interval I (the toplogy is the induced topology in $[0,1]$) of t such that $\gamma(s)\in D_t$ for all $s\in I$. Hence, by the definition of analytic continuation along γ [Def 2.2 on p.214], it follows that $f_s(z)=f_t(z)$ for $z\in D_t\cap D_s$ and hence $h(f_s(z))=h(f_t(z))=z$. Since both h and f_s are analytic on the region D_s, it follows that

$$h(f_s(z))=z \text{ for all } z\in D_s.$$

Hence, $I \subseteq T$. T is open. Moreover, for $t_n \in T$ such that $t_n \to t$. There exists $N > 0$ such that $\gamma(t_n) \in D_t$ for $n \geq N$. By the same idea of arguing T being open, it follows that $f_{t_N}(z) = f_t(z)$ for all $z \in D_{t_N} \cap D_t$, and hence $h(f_t(z)) = h(f_{t_N}(z)) = z$. By the fact that h ad f_t are analytic on D_t, we have

$$h(f_t(z)) = z \text{ for all } z \in D_{t.}.$$

Therefore $t \in T$ and T is closed. In conclusion, $T = [0, 1]$ because T is non-empty open and closed subset in $[0, 1]$.