Exercise. Q3. on p.201

**Lemma 0.1.** Suppose $O$ is an open set in $\mathbb{C}_\infty$ and $Q \subseteq O$. Let $D$ and $D'$ be the connected components of $O$ and $Q$ respectively. If $D \cap D' \neq \emptyset$, then $D' \subseteq D$.

**Proof.** Suppose $D' \not\subseteq D$. There exists $z \in D' \cap \partial D \subseteq Q \subseteq O$. As $O$ is open, there exists $\epsilon > 0$ such that a connected open ball $B_\epsilon(z) \subseteq O$. It follows that $z \in D \cap B_\epsilon(z)$ and hence, $D \cup B_\epsilon(z)$ is connected in $O$. Since $D$ is open which mean $z \notin D$. Therefore, $D \cup B_\epsilon(z) \supseteq D$. Contradiction. $\blacksquare$

**Lemma 0.2.** Let $D$ be the connected component of $\mathbb{C}_\infty - K$. Then $\partial D \subseteq \partial K$.

**Proof.** Suppose $\partial D \not\subseteq \partial K$. Then there exists $z \in \partial D \cap K^c$. Moreover, there exists $\epsilon > 0$ such that $B_\epsilon(z) \subseteq K^c$ because $K^c$ is open. Since $B_\epsilon(z) \cap D \neq \emptyset$ and $D$ is a connected component in $K^c$. We have $B_\epsilon(z) \subseteq D$ which mean $D \cap \partial D \neq \emptyset$. Contradiction as $D$ is open (Note $D$ is a connected component of a open set $K^c$.)

**Proof.** of Q3.

Let $D$ be a connected component of $\mathbb{C}_\infty - K$.

Case 1. $D$ is unbounded. Let $D'$ be the unbounded component of $\mathbb{C}_\infty - G \subseteq \mathbb{C}_\infty - K$. It follows that $\infty \in D \cap D'$. Therefore, by using Lemma 0.1, we have $D' \subseteq D$.

Case 2. $D$ is bounded. By the assumption that $\overline{D} \cap \partial G \neq \emptyset$, we let $z \in \overline{D} \cap \partial G$. If $z \in \partial D$, then by Lemma 0.2 we have $\partial K \cap \partial G \neq \emptyset$ which contradicts the fact that a compact $K \subseteq G$ where $G$ is open. Therefore, $z \in D \cap \partial G$. Since $D$ is an open connected component of $K^c$, there exists $\epsilon > 0$ such that $B_\epsilon(z) \subseteq D$ and hence $B_\epsilon(z) \subseteq D$. Using the fact that $B_\epsilon(z) \cap G \neq \emptyset$, we have $\partial D \neq \emptyset$ and from Lemma 0.1 $D$ contains a connected component of $G$. $\blacksquare$

Exercise. Q4. on p.201

From (b) to (a), by using Q3 on p.201, $\mathbb{C}_\infty - G$ meets every components of $\mathbb{C}_\infty - K$. For any $f$ which is analytic on a neighborhood of $K$ and $\epsilon > 0$, by Runge’s Theorem, there exists rational function $g(z)$ whose poles are in $\mathbb{C}_\infty - G$ such that

$$\sup_{z \in K} |f(z) - g(z)| < \epsilon.$$ 

Then (a) follows because $g \in H(G)$.

From (c) to (b), suppose b is false. There exists a connected component $D$ of $G$ such that $\overline{D} \cap \partial G = \emptyset$. Then $\partial D \subseteq \partial K$ and $\partial D \neq \emptyset$[otherwise, $D$ is both open and closed in $\mathbb{C}$]. For any $f \in H(G)$ and $z \in D$, we have

$$|f(z)| \leq \sup_{w \in \overline{D}} |f(w)| = \sup_{w \in \partial D} |f(w)| \leq \sup_{w \in \partial K} |f(w)| \leq \sup_{w \in K} |f(w)|$$

where the equality is by maximum modulus principle [Thm 1.2]. Contradicting with the condition in c.
From (a) to (c), let $z_0 \in G - K$. Let $f(z) = \frac{1}{z - z_0}$. Then by using $a$, for any $n \in \mathbb{N}$, there exists $f_n \in H(G)$ such that $\sup_{z \in K} |f(z) - f_n(z)| \leq \frac{1}{n}$. Hence, for all $z \in K$, it follows

$$|1 - (z - z_0) f_n(z)| \leq \frac{|z - z_0|}{n} \leq \frac{d(z_0, \partial K)}{n}.$$ 

Take $N$ large enough such that $\frac{d(z_0, \partial K)}{N} \leq \frac{1}{2}$. Let $g(z) = 1 - (z - z_0) f_N(z) \in H(G)$. Then, $|g(z_0)| = 1$ while $\sup_{z \in K} |g(z)| \leq \frac{1}{2}$.

Exercise. Q1 on p.213

(a) $G$ is path connected. Indeed for any two points $w_1, w_2 \in G$, we have two straight lines path $[a, w_1]$ and $[a, w_2] \subseteq G$. Then, $[w_1, a] \cup [a, w_2]$ is a path connecting $w_1$ and $w_2$ lying in $G$. [It is Alice Chan’s idea] To $G$ be open, for any $w \in G$, let $\epsilon = d([a, w], \gamma)$. For any $\hat{w} \in B_{\epsilon/2}(w)$, for any $u \in [a, \hat{w}]$, it follows that

$$d(u, [a, w]) \leq d(\hat{w}, w) \leq \epsilon/2.$$ 

Therefore, $u \notin \gamma$ and $\hat{w} \in G$. $G$ being open follows immediately.

It suffices to show that $\partial G = \gamma$ in the second part. It is clear that $\gamma \subseteq \partial G$ because for any $z \in \gamma$, then we can pick $\{z_n\} \subseteq [a, z]$ such that $z_n \to z$ and $z_n \in G$. For the other inclusion, for $r \in \partial G$, $[a, r] \cap \gamma \neq \emptyset$. Let $\ell : \mathbb{C} \to \mathbb{C}$ such that $\ell(z) = [a, z, \infty) \cap \gamma$ where $[a, z, \infty)$ is an unbounded line which passing from $a$ through $z$. Hence, $\ell(r) \in [a, r]$. Suppose $\ell(r) \neq r$, i.e. $r \notin \gamma$. By the continuity of $\ell$, there exists an open ball $B_r$ of $r$ such that $\ell(\hat{r}) \in [a, \hat{r}]$ and $\ell(\hat{r}) \neq \hat{r}$ for $\hat{r} \in B_r$. It follows that $r \notin \partial G$ because there does not exists $w_n \in G$ such that $w_n \to r$. Contradiction.

(b) W.L.O.G., we assume $\gamma : [0, 1] \to \mathbb{C}$. For any $\epsilon > 0$, there exists $\delta > 0$ such that

$$|f(z) - f(w)| < \epsilon \text{ if } z, w \in G \text{ and } |z - w| \leq \delta + a\delta.$$ 

We denote a simple closed rectifiable curve $\hat{\gamma} [0, 1]$ lying in $G$ as

$$\hat{\gamma}(t) = a + (1 - \delta)(\gamma(t) - a). \quad (0.1)$$

Then, $|f(\gamma(t) - \hat{\gamma}(t))| < \epsilon$ and $\int_\gamma f = 0$. Let $0 < t_1 < \cdots < t_m < 1$ be a partition of $[0, 1]$, $\gamma_k := \gamma(t_k)$ and $\hat{\gamma}_k := \hat{\gamma}(t_k)$ such that

$$I = \left| \int_\gamma f - \sum_{k=1}^m f(\gamma_k) [\gamma_k - \gamma_{k-1}] \right| < \epsilon$$

and

$$II = \left| \int_{\hat{\gamma}} f - \sum_{k=1}^m f(\hat{\gamma}_k) [\hat{\gamma}_k - \hat{\gamma}_{k-1}] \right| < \epsilon.$$
Then,
\[
\left| \int_{\gamma} f \right| = \left| \int_{\gamma} f - \int_{\hat{\gamma}} f \right| \\
\leq I + II + \sum_{k=1}^{m} f(\gamma_k) (\gamma_k - \gamma_{k-1}) - f(\hat{\gamma_k}) (\hat{\gamma_k} - \hat{\gamma}_{k-1}) \\
\leq 2\epsilon + \sum_{k=1}^{m} |f(\gamma_k) - f(\hat{\gamma_k})| |\gamma_k - \gamma_{k-1}| + \\
\sum_{k=1}^{m} |f(\hat{\gamma_k})| (|\gamma_k - \gamma_{k-1}| - |\hat{\gamma_k} - \hat{\gamma}_{k-1}|).
\]

Let $V(\gamma)$ be the total variation of $\gamma$. Then, it can be seen that
\[
\sum_{k=1}^{m} |f(\gamma_k) - f(\hat{\gamma_k})| |\gamma_k - \gamma_{k-1}| \leq \epsilon V(\gamma).
\]

Let $M = \max_{t \in [0,1]} |\gamma(t)|$. By using Eq. \((0.1)\), it follows that
\[
\sum_{k=1}^{m} |f(\hat{\gamma_k})| (|\gamma_k - \gamma_{k-1}| - |\hat{\gamma_k} - \hat{\gamma}_{k-1}|) \leq M \sum_{k=1}^{m} \delta |\gamma_k - \gamma_{k-1}| \leq M\epsilon V(\gamma).
\]

Therefore,
\[
\left| \int_{\gamma} f \right| \leq 2\epsilon + \epsilon V(\gamma) + M\epsilon V(\gamma)
\]
and $\left| \int_{\hat{\gamma}} f \right| = 0$ is followed.

(c) $\overline{G}$ is bounded because for any $z, w \in \overline{G}$, $|z - w| < |z - a| + |w - a| < 2 \text{dist}(a, \gamma) < \infty$.

Therefore, if $z \notin \overline{G}$, then $z$ lies in the unbounded component of $\mathbb{C} - \gamma$ and hence $n(\gamma, z) = 0$ by Theorem 4.4 on p.82. For $z \in G$ where $G$ is a region and $a \in G$. it follows that $n(\gamma, z) = n(\gamma, a)$ by Theorem 4.4 on p.82. Let $\epsilon > 0$ such that $a + \epsilon e^{2\pi i t} \in G$ for $t \in [0, 1]$. Then, there exists a strictly monotone function $\sigma(t)$ from $[0, 1]$ to $[0, 1]$ such that $\gamma(t) = \ell (a + \epsilon e^{2\pi i t})$ for $t \in [0, 1]$. Let
\[
\hat{\gamma}(t) = a + \epsilon e^{2\pi i t}.
\]

Let
\[
F(t, \tau) = \tau \gamma(t) + (1 - \tau) \hat{\gamma}(t).
\]

It can be verified that $\gamma$ is homotopic to $\hat{\gamma}$. Hence $n(\gamma, a) = n(\hat{\gamma}, a) = \pm 1$.

**Exercise. Q7. on p.217**

First, we note that $0 \in T$ because $h(f_0(z)) = z$ for $z \in D_0$ by the assumption. Moreover, suppose $t \in T$, then there exists an open interval $I$ (the topology is the induced topology in $[0,1]$) of $t$ such that $\gamma(s) \in D_I$ for all $s \in I$. Hence, by the definition of analytic continuation along $\gamma$ [Def 2.2 on p.214], it follows that $f_s(z) = f_t(z)$ for $z \in D_t \cap D_s$ and hence $h(f_s(z)) = h(f_t(z)) = z$. Since both $h$ and $f_s$ are analytic on the region $D_s$, it follows that
\[
h(f_s(z)) = z \text{ for all } z \in D_s.
Hence, $I \subseteq T$. $T$ is open. Moreover, for $t_n \in T$ such that $t_n \to t$. There exists $N > 0$ such that $\gamma(t_n) \in D_t$ for $n \geq N$. By the same idea of arguing $T$ being open, it follows that $f_{t_N}(z) = f_t(z)$ for all $z \in D_{t_N} \cap D_t$, and hence $h(f_t(z)) = h(f_{t_N}(z)) = z$. By the fact that $h$ and $f_t$ are analytic on $D_t$, we have

\[ h(f_t(z)) = z \text{ for all } z \in D_t. \]

Therefore $t \in T$ and $T$ is closed. In conclusion, $T = [0, 1]$ because $T$ is non-empty open and closed subset in $[0, 1]$. 
