HW4

Exercise. Q2. on p.221

(a) By Proposition 2.4 on p.215, it follows that $[g]_{\gamma(t)} = [f]_{\gamma(t)}$. Therefore, there exists an open set U of $\gamma(t)$ such that $g = f$ on U. Since f is analytic on $B(\gamma(t), R(t))$. By the unique analytic extension, we can also conclude that g is analytic on $B(\gamma(t), r(t))$. Hence, $R(t) \leq r(t)$. Similarly, we can also verified that $r(t) \leq R(t)$. Therefore, $R(t) = r(t)$.

(b) Since $\gamma(t)$ lies in $\mathbb{C} - \{x \in \mathbb{R} : x \leq 0\}$ on which the principal branch log can be defined, let f_t be the principal branch log and $D_t = D(\gamma(t), 1 + at)$ where $1 + at$ is a distance between $\gamma(t)$ and the origin). Then (f_t, D_t) is an analytic continuation of (f_0, D_0) along $\gamma(t)$ where $f_0 = f$ and $D_0 = B(1; 1)$. By part a, we have $R(t) = 1 + at$.

(c) It can be verified that $f_t = \ln r + i\theta$ where $-\pi + 2\pi t < \theta < \pi + 2\pi t$ and $D_t = (\gamma(t), 1 - at)$, where $1 - at$ is a distance between $\gamma(t)$ and the origin, is an analytic continuation of (f_0, D_0) along $\gamma(t)$. By part a, we have $R(t) = 1 - at$.

(d) In part b, let $F(a) = \min\{R(t) : t \in [0, 1]\} = 1$. Hence, $\lim_{a\to 0} F(a) = \lim_{a\to\infty} F(a) = 1$. In part c, let $F(a) = \min\{R(t) : t \in [0, 1]\} = 1 - a$. Hence, $\lim_{a\to 0} F(a) = 1$ and $\lim_{a\to 1^-} F(a) = 0$.

Exercise. Q1. on p.233

Given $(z, [f]_z) \in \mathscr{S}(\mathbb{C})$, let $B(f(z), \epsilon)$ be an open ball of $f(z)$ with radius ϵ. It suffices to check that $F^{-1}(B(f(z), \epsilon))$ is open in $\mathscr{S}(\mathbb{C})$. For any $(a, [g]_a) \in F^{-1}(B(f(z), \epsilon))$, it follows that $g(a) \in B(f(z), \epsilon)$ which means there exists $\epsilon_a > 0$ such that

$$|f(z) - g(a)| = \epsilon_a < \epsilon.$$

Since g is analytic locally at a, there exists $\delta > 0$ such that $|g(a) - g(b)| < \epsilon - \epsilon_a$ if $|a - b| < \delta$. Then, for $b \in B(a, \delta)$, it follows that

$$|f(z) - g(b)| = |f(z) - g(a)| + |g(a) - g(b)| < \epsilon.$$

and hence, $(b, [g]_b) \in F^{-1}(B(f(z), \epsilon))$. Therefore, we can conclude that

$$N(g, B(a, \delta)) \subseteq F^{-1}(B(f(z), \epsilon))$$

and $F^{-1}(B(f(z), \epsilon))$ being open follows.

Exercise. Q2. on p.233

We let $G_1 = \mathbb{C} - \{z \in \mathbb{C} : \operatorname{Re}(z) \leq 0\}$ and F be a principal branch of logarithm on G_1. Then,

$$F(z) := \log|z| + i\arg(z)$$

where $-\pi < \arg(z) < \pi$. Then $\mathscr{F}$ is a complete of analytic function where the function element is (F, G_1).

Given$a \in D$, we first show $[f]_a \in \mathscr{F}$. Since f is a branch of the logarithm on D, there exists $\phi \in \mathbb{R}$ such that

$$f(z) := \log|z| + i\arg(z)$$

where $-\pi + \phi < \arg(z) < \pi + \phi$. Let $\varphi_a = \arg(a)$ where $-\pi + \phi < \arg(a) < \pi + \phi$. Then, let $b = |a| \in G_1$ and $\gamma(t) = |a| e^{2\pi i t}$ where $t \in [0, \varphi_a]$ be a path from b to a. It can be verified that $[f]_a$ is the analytic continuation of $[f]_b$ along γ. Hence, $[f]_a \in \mathscr{F}$.

Then, we show $[f]_a \in \mathscr{F}$ implies $f : D \to \mathbb{C}$ is a branch of the logarithm. It suffices to check $\exp(f(z)) = z$ for $z \in D$. By using the condition that $[f]_a \in \mathscr{F}$, there exists a path $\gamma(t)$ from $\gamma(0) = b \in G_1$ to $\gamma(1) = a \in D$ such that $[f]_a$ is an analytic continuation of $[F]_b$. It can be verified that $\exp(F(z)) = z$ on G_1. Then, by using Q7 on p.217 with $h(z) = \exp(z)$ and $G = \mathbb{C}$ in Q7, it follows that $\exp(f(z)) = z$ on D.