Exercise. Q2. on p.221

(a) By Proposition 2.4 on p.215, it follows that \([g]_{\gamma(t)} = [f]_{\gamma(t)}\). Therefore, there exists an open set \(U\) of \(\gamma(t)\) such that \(g = f\) on \(U\). Since \(f\) is analytic on \(B(\gamma(t), R(t))\). By the unique analytic extension, we can also conclude that \(g\) is analytic on \(B(\gamma(t), r(t))\). Hence, \(R(t) \leq r(t)\). Similarly, we can also verify that \(r(t) \leq R(t)\). Therefore, \(R(t) = r(t)\).

(b) Since \(\gamma(t)\) lies in \(\mathbb{C} - \{x \in \mathbb{R} : x \leq 0\}\) on which the principal branch \(\log\) can be defined, let \(f_t\) be the principal branch \(\log\) and \(D_t = D(\gamma(t), 1 + at)\) where \(1 + at\) is a distance between \(\gamma(t)\) and the origin. Then \((f_t, D_t)\) is an analytic continuation of \((f_0, D_0)\) along \(\gamma(t)\) where \(f_0 = f\) and \(D_0 = B(1; 1)\). By part a, we have \(R(t) = 1 + at\).

(c) It can be verified that \(f_t = \ln r + i\theta\) where \(-\pi + 2\pi t < \theta < \pi + 2\pi t\) and \(D_t = (\gamma(t), 1 - at)\), where \(1 - at\) is a distance between \(\gamma(t)\) and the origin, is an analytic continuation of \((f_0, D_0)\) along \(\gamma(t)\). By part a, we have \(R(t) = 1 - at\).

(d) In part b, let \(F(a) = \min\{R(t) : t \in [0, 1]\}\) = 1. Hence, \(\lim_{a\to0} F(a) = \lim_{a\to\infty} F(a) = 1\). In part c, let \(F(a) = \min\{R(t) : t \in [0, 1]\}\) = 1 - a. Hence, \(\lim_{a\to0} F(a) = 1\) and \(\lim_{a\to\infty} F(a) = 0\).

Exercise. Q1. on p.233

Given \((z, [f]_z) \in \mathcal{S}(\mathbb{C})\), let \(B(f(z), \epsilon)\) be an open ball of \(f(z)\) with radius \(\epsilon\). It suffices to check that \(F^{-1}(B(f(z), \epsilon))\) is open in \(\mathcal{S}(\mathbb{C})\). For any \((a, [g]_a) \in F^{-1}(B(f(z), \epsilon))\), it follows that \(g(a) \in B(f(z), \epsilon)\) which means there exists \(\epsilon_a > 0\) such that

\[|f(z) - g(a)| = \epsilon_a < \epsilon.\]

Since \(g\) is analytic locally at \(a\), there exists \(\delta > 0\) such that \(|g(a) - g(b)| < \epsilon - \epsilon_a\) if \(|a - b| < \delta\). Then, for \(b \in B(a, \delta)\), it follows that

\[|f(z) - g(b)| = |f(z) - g(a)| + |g(a) - g(b)| < \epsilon.\]

and hence, \((b, [g]_b) \in F^{-1}(B(f(z), \epsilon))\). Therefore, we can conclude that \(N(g, B(a, \delta)) \subseteq F^{-1}(B(f(z), \epsilon))\) and \(F^{-1}(B(f(z), \epsilon))\) being open follows.

Exercise. Q2. on p.233

We let \(G_1 = \mathbb{C} - \{z \in \mathbb{C} : \Re(z) \leq 0\}\) and \(F\) be a principal branch of logarithm on \(G_1\). Then,

\[F(z) := \log |z| + i \arg(z)\]

where \(-\pi < \arg(z) < \pi\). Then \(\mathcal{S}\) is a complete of analytic function where the function element is \((F, G_1)\).

Given \(a \in D\), we first show \([f]_a \in \mathcal{S}\). Since \(f\) is a branch of the logarithm on \(D\), there exists \(\phi \in \mathbb{R}\) such that

\[f(z) := \log |z| + i \phi(z)\]
where \(-\pi + \phi < \text{arg} (z) < \pi + \phi\). Let \(\varphi_a = \text{arg} (a)\) where \(-\pi + \phi < \text{arg} (a) < \pi + \phi\).

Then, let \(b = |a| \in G_1\) and \(\gamma (t) = |a| e^{2\pi i t}\) where \(t \in [0, \varphi_a]\) be a path from \(b\) to \(a\). It can be verified that \([f]_a\) is the analytic continuation of \([f]_b\) along \(\gamma\). Hence, \([f]_a \in \mathcal{F}\).

Then, let \([f]_a \in \mathcal{F}\) implies \(f : D \to \mathbb{C}\) is a branch of the logarithm. It suffices to check \(\exp (f (z)) = z\) for \(z \in D\). By using the condition that \([f]_a \in \mathcal{F}\), there exists a path \(\gamma (t)\) from \(\gamma (0) = b \in G_1\) to \(\gamma (1) = a \in D\) such that \([f]_a\) is an analytic continuation of \([F]_b\). It can be verified that \(\exp (F (z)) = z\) on \(G_1\).

Then, by using Q7 on p.217 with \(h (z) = \exp (z)\) and \(G = \mathbb{C}\) in Q7, it follows that \(\exp (f (z)) = z\) on \(D\).