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PID: ____________________
Discussion Section: ____________________

Math 109 : Midterm 2
Introduction to Mathematical Reasoning

You have 1 hour and 20 minutes. You may not use any electronic devices, books, notes.

For every exercise you are given two pages, the second of which you may use for rough work.

1. (20 points) Prove or disprove the following statements:

(a) (10 points) $\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, y - x \geq 0$.

(b) (10 points) $(\exists a \in \mathbb{Z}, n = 2a+1) \Rightarrow (\exists b \in \mathbb{Z}, n^2 = 4b+1)$.

(a) False

$\forall x \in \mathbb{R}$, $y = x - 1 < x$.

(b) True

If $\exists a \in \mathbb{Z}$ s.t. $n = 2a+1$, then

$$n^2 = (2a+1)^2 = 4a^2 + 4a + 1 = 4\underbrace{(a^2+a)}_{= \, b \in \mathbb{Z}} + 1.$$

2. (25 points) Let $f : \mathbb{Z} \to \mathbb{Z}$ and $g : \mathbb{Z} \to \mathbb{Z}$ be given by $f(x) = x^2$ and $g(x) = 3x + 1$.

(a) (5 points) Find the functions $f \circ g$ and $g \circ f$.

(b) (10 points) Determine and prove whether $g \circ f$ is surjective.

(c) (10 points) Prove that $f \circ g$ is injective.

(a) $f \circ g : \mathbb{Z} \to \mathbb{Z} \qquad x \mapsto (3x+1)^2$

$g \circ f : \mathbb{Z} \to \mathbb{Z} \qquad x \mapsto 3x^2+1$

(b) Observe $\forall x \in \mathbb{Z}$, $(g \circ f)(x) = 3x^2 + 1 \geq 1$.

So $0 \notin \mathrm{Im}(g \circ f)$.

Hence $g \circ f$ is not surjective.

(c) Let $x_1, x_2 \in \mathbb{Z}$. Assume $(f \circ g)(x_1) = (f \circ g)(x_2)$,

i.e. $(3x_1+1)^2 = (3x_2+1)^2$.

$\Leftrightarrow \quad (3x_1+1)^2 - (3x_2+1)^2 = 0$

$\Leftrightarrow \quad (3x_1+1+3x_1+1)(3x_1+1-3x_1-1) = 0$

$\Leftrightarrow \quad 3(x_1 - x_2)(3(x_1+x_2)+2) = 0$

$\Leftrightarrow \quad x_1 = x_2 \quad \text{or} \quad 3(x_1+x_2) = -2$

Since $x_1 + x_2 \in \mathbb{Z}$ and $3 \nmid 2$, $3(x_1+x_2) \neq -2$.

Hence $x_1 = x_2$. ~~and~~ ~~$g \circ f$~~

Therefore $f \circ g$ is injective.

3. (15 points) Suppose that $f : X \to Y$ and $g : Y \to Z$ are functions between sets. Assume that $g \circ f : X \to Z$ is an injection. Prove that f is an injection.

Let $x_1, x_2 \in X$ and suppose $f(x_1) = f(x_2)$.

Then $(g \circ f)(x_1) = g(f(x_1)) = g(f(x_2)) = (g \circ f)(x_2)$.

Since $g \circ f$ is an injection, $x_1 = x_2$.

$\square$.

4. (20 points) Let X be a non-empty finite set and $n \geq 1$ be an integer.

(a) (5 points) State what it means for the cardinality of X to be n.

(b) (15 points) Assume that Y is a non-empty finite set such that $|Y| = |X|$. Prove that there exists a bijection $X \to Y$.

(a) $\exists$ a bijection $f: \mathbb{N}_n \to X$.

(b) Since $0 < |X|, |Y| < \infty$, suppose $|X| = |Y| = n > 0$.

$|X| = n \Rightarrow \exists$ bijection $f: \mathbb{N}_n \to X$.

$|Y| = n \Rightarrow \exists$ bijection $g: \mathbb{N}_n \to Y$.

Then $g \circ f^{-1}: X \to \mathbb{N}_n \to Y$ is a bijection b/c f^{-1} is a bijection and composition of bijections is a bijection.

$\square$

5. (20 points) Let X, Y and Z be subsets of $\mathbb{N}_n = \{1, 2, ..., n\}$, for some integer $n \geq 1$.

(a) (10 points) Assume that $|X| + |Y| > n$.
Prove that $X \cap Y \neq \emptyset$.

(b) (10 points) Assume that $|X| + |Y| + |Z| > 2n$.
Prove that $X \cap Y \cap Z \neq \emptyset$.

Hint: For both parts, use the inclusion-exclusion principle and that $|A| \leq n$, for any subset A of $\mathbb{N}_n$. For part (b), find a way to apply part (a).

(a) $X, Y \subset \mathbb{N}_n \Rightarrow X \cup Y \subset \mathbb{N}_n \Rightarrow |X \cup Y| \leq n$.

By inclusion / exclusion principle,

$n \geqslant |X \cup Y| = |X| + |Y| - |X \cap Y| > n - |X \cap Y|$.

So $|X \cap Y| > 0$.

Hence $X \cap Y \neq \phi$.

(b) It suffices to show $|X| + |Y \cap Z| > n$ by part (a).

Since $|Y \cup Z| = |Y| + |Z| - |Y \cap Z|$,

$|Y| + |Z| = |Y \cup Z| + |Y \cap Z|$.

So, $|X| + |Y \cup Z| + |Y \cap Z| > 2n$.

Thus $|X| + |Y \cap Z| > 2n - |Y \cup Z| \geqslant n$

b/c $|Y \cup Z| \leq n$.

$\square$.