1. Show that for any matrix $A \in \mathbb{R}^{m \times n}$ ($n > m$) there is a nonzero vector $x \in \mathbb{R}^n$ such that $Ax = 0$.

**Solution:** We are going to prove the statement using induction by $n$.

The base case is for $n = 2$. The matrix $A$ should be a $1 \times 2$ matrix with entries $a, b \in \mathbb{R}$.

- If $b = 0$, then $a \cdot 0 + b \cdot 1 = 0$ in other words $x = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is a solution for the equation $Ax = 0$.

- Otherwise, consider $x = \begin{bmatrix} 1 \\ -a/b \end{bmatrix}$ and note that $Ax = 0$.

So we proved the base case.

Now we need to prove the induction step from $k$ to $k + 1$. The induction hypothesis is that for any $m < k$ and any matrix $B \in \mathbb{R}^{m \times k}$ there is a nonzero vector $x \in \mathbb{R}^n$ such that $Bx = 0$. Assume we are given a matrix $A \in \mathbb{R}^{m \times (k+1)}$, where $m < k + 1$.

- If the last column of $A$ is equal to 0 consider $x = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$ and note that $Ax = 0$.

- Otherwise, WLOG we may assume that $A_{m,k+1} \neq 0$. Note that the last equation is $A_{m,1}x_1 + \cdots + A_{m,k}x_k + A_{m,k+1}x_{k+1} = 0$. Hence, if we substitute $x_{k+1} = \frac{-A_{m,1}}{A_{m,k+1}}x_1 + \cdots + \frac{-A_{m,k}}{A_{m,k+1}}x_k$ we get a system of $m - 1$ equations and $k$ variables which has a nonzero solution $x_1, \ldots, x_k$ by the induction hypothesis. As a result, the vector $x = \begin{bmatrix} x_1 \\ \vdots \\ x_k \\ \frac{-A_{m,1}}{A_{m,k+1}}x_1 + \cdots + \frac{-A_{m,k}}{A_{m,k+1}}x_k \end{bmatrix}$ is a nonzero solution of the equation $Ax = 0$. 


2. Show that all the elements of \( \{0, 1\}^n \) (Binary strings) may be ordered such that every successive strings in this order are different only in one character.

For example, for \( n = 2 \) the order may be 00, 01, 11, 10.

**Solution:** Let us construct these orders inductively. The order for \( n = 2 \) is given in the problem statement. Assume that the order for \( n \) is \( g_{n,1}, \ldots g_{n,k_n} \). We define that the order for \( n + 1 \) as follows

\[
g_{1,0}, \ldots, g_{k_n,0}, g_{k_n,1}, \ldots, g_{1,1},
\]

where \( g_i b \) is \( g_i \) with appended \( b \) at the end.

Now we need to prove that the defined sequence satisfies the constraint. The base case for \( n = 2 \) is clear. Let us prove the induction step from \( n \) to \( n + 1 \). By induction hypothesis the order \( g_{n,1}, \ldots g_{n,k_n} \) satisfies the constraint. Note that \( g_i 0 \) differs with \( g_{i+1} 0 \) only in one symbol (by the induction hypothesis) and \( g_{k_n} 0 \) and \( g_{k_n} 1 \) are different only in the last symbol. Additionally, it is easy to see that we ordered all the strings. Hence, we proved that such an ordering is possible for every \( n \).