1. (10 points) Show that the set \( \{0, 1\} \times [n] \) has cardinality \( 2^n \).

**Solution:** By definition of cardinality, to prove that cardinality of \( \{0, 1\} \times [n] \) is equal to \( 2^n \) we need to prove that there is a bijection \( f : [2n] \to \{0, 1\} \times [n] \).

Consider the function

\[
    f(i) = \begin{cases} 
    (0, i) & i \leq n \\
    (1, i - n) & n < i \leq 2n 
    \end{cases}
\]

Note that the function is defined correctly since the first coordinate is always 0 or 1 and the second coordinate is always from \([n]\) due to the fact that if \( i \leq n \), then \( i \in [n] \) and if \( i > n \), then \( i - n \in [n] \).

We are going to prove now that it is a bijection.

**injection:** Assume the opposite i.e. that there are \( i_0 \neq i_1 \) such that \( f(i_0) = f(i_1) \). Without loss of generality we may assume that \( i_0 < i_1 \). Consider three cases:

1. \( (i_0, i_1 \leq n) \) Note that \( (0, i_0) = f(i_0) \) and \( f(i_1) = (0, i_1) \). So \( i_0 = i_1 \) since \( f(i_0) = f(i_1) \), which is a contradiction.

2. \( (i_0, i_1 > n) \) Note that \( (1, i_0 - n) = f(i_0) \) and \( f(i_1) = (1, i_1 - n) \). So \( i_0 = i_1 \) since \( f(i_0) = f(i_1) \), which is a contradiction.

3. \( (i_0 \leq n < i_1) \) Note that \( (0, i_0) = f(i_0) \) and \( f(i_1) = (1, i_1 - n) \). So \( 0 = 1 \) since \( f(i_0) = f(i_1) \), which is a contradiction.

**surjection:** Let us fix some \( (a, b) \in \{0, 1\} \times [n] \). We need to show that it belongs to \( \text{Im} f \).

1. \( (a = 0) \) Note that \( b \leq n \), hence, \( f(b) = (0, b) = (a, b) \). So \( (a, b) \in \text{Im} f \).

2. \( (a = 1) \) Note that \( 2n \geq n + b > n \), hence, \( f(n + b) = (1, (n + b) - b) = (1, b) = (a, b) \). So \( (a, b) \in \text{Im} f \).
2. (10 points) Let us consider group theory, it is a theory with undefined terms: group-element and times. 
(if $a$ and $b$ are group elements, we denote $a$ times $b$ by $a \cdot b$), and axioms:

1. $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for every group-elements $a$, $b$, and $c$;
2. there is a unique group-element $e$ such that $e \cdot a = a \cdot e$ for every group-element $a$ (we say that such an element is the identity element);
3. for every group-element $a$ there is a group-element $b$ such that $a \cdot b = e$, where $e$ is the identity element;
4. for every group-element $a$ there is a group-element $b$ such that $b \cdot a = e$, where $e$ is the identity element.

Let $e$ be the identity element. Show the following statements

- if $b_0 \cdot a = b_1 \cdot a = e$, then $b_0 = b_1$, for every group-elements $a$, $b_0$, and $b_1$.
- if $a \cdot b_0 = a \cdot b_1 = e$, then $b_0 = b_1$, for every group-elements $a$, $b_0$, and $b_1$.
- if $a \cdot b_0 = b_1 \cdot a = e$, then $b_0 = b_1$, for every group-elements $a$, $b_0$, and $b_1$.

**Solution:**

- Assume that $b_0 \cdot a = b_1 \cdot a = e$ for some group-elements $b_0$, $b_1$, and $a$. By axiom 3, there is $b$ such that $a \cdot b = e$. If we multiply all sides of the equality by $b$, we get $(b_0 \cdot a) \cdot b = (b_1 \cdot a) \cdot b = e \cdot b$. By axiom 1, it implies $b_0 \cdot (a \cdot b) = b_1 \cdot (a \cdot b) = e \cdot b$. Hence, by the definition of $b$, $b_0 \cdot e = b_1 \cdot e = e \cdot b$. As a result, by axiom 2, $b_0 = b_1 = b$.

- Assume that $a \cdot b_0 = a \cdot b_1 = e$ for some group-elements $b_0$, $b_1$, and $a$. By axiom 3, there is $b$ such that $b \cdot a = e$. If we multiply all sides of the equality by $b$, we get $b \cdot (a \cdot b_0) = b \cdot (a \cdot b_1) = b \cdot e$. By axiom 1, it implies $(b \cdot a) \cdot b_0 = (b \cdot a) \cdot b_1 = b \cdot e$. Hence, by the definition of $b$, $e \cdot b_0 = e \cdot b_1 = b \cdot e$. As a result, by axiom 2, $b_0 = b_1 = b$.

- Finally, assume that $a \cdot b_0 = b_1 \cdot a = e$ for some group-elements $b_0$, $b_1$, and $a$. Let us multiply $a \cdot b_0 = e$ by $b_1$, we get $b_1 \cdot (a \cdot b_0) = b_1 \cdot e$. By axiom 1, $(b_1 \cdot a) \cdot b_0 = b_1 \cdot e$. By axiom 2, $(b_1 \cdot a) \cdot b_0 = b_1$. Hence, by the assumption $e \cdot b_0 = b_1$ and by axiom 2, $b_0 = b_1$. 