1. (10 points) Let $A_1, \ldots, A_\ell$ be different subsets of $[n]$ such that $A_i \cap A_j \neq \emptyset$ for all $i \neq j \in [\ell]$. Prove that $\ell \leq 2^n - 1$.

**Solution:** We are going to prove this statement using the pigeonhole principle. The main component of the proof is the fact that $A \cap \bar{A} = \emptyset$ for any set $A \subseteq [n]$, where $\bar{A} = [n] \setminus A$.

Let $L = \{A_1, \ldots, A_\ell\}$ and $R = \{\{A, \bar{A}\} \mid A \subseteq A\}$. It is easy to see that $|R| = 2^{n-1}$. Define the function $f : L \to R$ such that $f(A_i) = \{A_i, \bar{A_i}\}$. Let us prove that $f$ is an injection; assume the opposite i.e. that there are $i \neq j$ such that $f(A_i) = f(A_j)$. Since $f(A_i) = f(A_j)$ it implies that $A_i = \bar{A_j}$ but it contradicts to the statement that $A_i \cap A_j = \bar{A_j} \cap A_j = \emptyset$.

Hence, there is an injection from $L$ to $R$ which implies that $|L| \leq |R|$. As a result, we proved that $\ell \leq 2^{n-1}$.
2. (10 points) How many pairs of subsets $A, B \subseteq [n]$ are there such that $A \cap B \neq \emptyset$.

**Solution:** We need to find the cardinality of the set $\{(A, B) \mid A, B \subseteq [n] \text{ and } A \cap B \neq \emptyset\}$. Note that by inclusion exclusion principle

$$|\{(A, B) \mid A, B \subseteq [n] \text{ and } A \cap B \neq \emptyset\}| = |\{(A, B) \mid A, B \subseteq [n] \text{ and } A \cap B \neq \emptyset\}| + |\{(A, B) \mid A, B \subseteq [n] \text{ and } A \cap B = \emptyset\}|.$$

Hence, if we find cardinalities of the sets $\{(A, B) \mid A, B \subseteq [n] \text{ and } A \cap B \neq \emptyset\}$ and $\{(A, B) \mid A, B \subseteq [n] \text{ and } A \cap B = \emptyset\}$ it will allow us to find the answer.

We will compute cardinalities of these sets in a similar manner.

- $\{(A, B) \mid A, B \subseteq [n]\}$ For each $i \in [n]$ there are four ways to put or not put it into sets $A$ and $B$:
  1. we may put $i$ to both of them,
  2. we may put $i$ to none of them,
  3. we may put $i$ only to $A$,
  4. we may put $i$ only to $B$.

Hence, by multiplicity law, there are $4^n$ ways to form $(A, B)$.

- $\{(A, B) \mid A, B \subseteq [n] \text{ and } A \cap B = \emptyset\}$ Similarly, for each $i \in [n]$ there are three ways to put or not put it into sets $A$ and $B$ such that $A \cap B = \emptyset$:
  1. we may not put $i$ to both of them,
  2. we may put $i$ only to $A$,
  3. we may put $i$ only to $B$.

Hence, by multiplicity law, there are $3^n$ ways to form $(A, B)$ such that $A \cap B = \emptyset$.

As a result, the answer is $4^n - 3^n$.

This proof would be enough, however, let us give a more formal explanation why the cardinality of the set $\{(A, B) \mid A, B \subseteq [n] \text{ and } A \cap B = \emptyset\}$ is $3^n$. To prove this completely formally we are going to construct a bijection from this set to $[3]^n$.

Consider $f : \{(A, B) \mid A, B \subseteq [n] \text{ and } A \cap B = \emptyset\} \to [3]^n$ such that $f(A, B) = (x_1, \ldots, x_n)$, where

$$x_i = \begin{cases} 
1 & \text{if } i \in A \text{ and } i \notin B \\
2 & \text{if } i \notin A \text{ and } i \in B \\
3 & \text{if } i \notin A \text{ and } i \notin B
\end{cases}$$

Note that the function is well defined due to the fact that $A \cap B = \emptyset$. We need to prove now that it is a bijection. To do this we can simply construct the inverse function $e : [3]^n \to \{(A, B) \mid A, B \subseteq [n] \text{ and } A \cap B = \emptyset\}$ such that $e(x_1, \ldots, x_n) = (A, B)$, where $A = \{i \in [n] \mid x_i = 1\}$ and $B = \{i \in [n] \mid x_i = 2\}$. Obviously $e$ is the inverse of $f$, hence, $f$ is a bijection and $|\{(A, B) \mid A, B \subseteq [n] \text{ and } A \cap B = \emptyset\}| = 3^n$. 