1. (50 points) Check all the correct statements (in this question only the answers will be graded).

- gcd(24, 18) = 6.
- The function $f : [-\frac{\pi}{2}, \frac{\pi}{2}] \to \mathbb{R}$ such that $f(x) = \arctan x$ is a bijection.
- The cardinality of the set $F(X, [3]) = (4^n)^3$, where $X = F([4], [n])$.
- The cardinality of the set $I([3], [n]) = n(n - 1)(n - 2)$.
- $\binom{10}{2} = 90$.

**Solution:**

1. Note that $D(24) = \{1, 2, 3, 4, 6, 8, 12, 24\}$ and $D(18) = \{1, 2, 3, 6, 9, 18\}$. Hence, $\gcd(24, 18) = 6$.

2. No it is not a bijection since $\arctan$ is increasing function, hence, the value of $\text{Im} f \subseteq [f(-\frac{\pi}{2}), f(\frac{\pi}{2})]$.

3. The cardinality of the set $X = F([4], [n])$ is equal to $n^4$, hence, the cardinality of the set $F(X, [3])$ is equal to $3n^4$.

4. $\binom{10}{2} = \frac{10 \cdot 9}{2 \cdot 1} = 45$. 


2. (a) (5 points) Let \( n \), \( a \), and \( b \) be some integers. Show that if two numbers \( a \) and \( b \) have the same reminders when divided by \( n \), then \( a - b \) is divisible by \( n \).

**Solution:** There are integers \( k \), \( \ell \) and \( r \) such that \( a = kn + r \) and \( a = \ell n + r \) since \( a \) and \( b \) have the same reminder when divided by \( n \).

Note that \( a - b = (k - \ell)n \), hence, is divisible by \( n \).

(b) (5 points) Prove that for every integers \( a_1, \ldots, a_n \) there are \( k > 0 \) and \( \ell \geq 0 \) such that \( k + \ell \leq n \) and \( \sum_{i=k}^{k+\ell} a_i \) is divisible by \( n \).

**Solution:** Let us consider the function \( f : \{0,1,\ldots,n\} \to \{0,1,\ldots,n-1\} \) such that \( f(i) \) is equal to the reminder of \( \sum_{j=1}^{i} a_j \) (if \( i < 1 \), the sum is equal to 0) when divided by \( n \). By the pigeonhole principle there are \( i_0 < i_1 \) such that \( f(i_0) = f(i_1) \); hence, \( f(i_1) - f(i_0) = \sum_{j=1}^{i_1} a_j - \sum_{j=1}^{i_0} a_j = \sum_{j=i_0+1}^{i_1} a_j \) is divisible by \( n \).
3. (10 points) We say that sets \( A_1, A_2, \) and \( A_3 \) are pairwise disjoint iff \( A_i \cap A_j = \emptyset \) for every \( i \neq j \in [3] \).

Construct a bijection from \( \{0, 1, 2, 3\}^n \) to \( \{(A, B, C) \mid A, B, C \subseteq [n] \text{ and } A, B, C \text{ are pairwise disjoint}\} \).

**Solution:** Let us consider the function \( f : \{0, 1, 2, 3\}^n \to \{(A, B, C) \mid A, B, C \subseteq [n] \text{ and } A, B, C \text{ are pairwise disjoint}\} \) such that \( f(x_1, \ldots, x_n) = (A_x, B_x, C_x) \), where \( A_x = \{i \in [n] \mid x_i = 1\} \), \( B_x = \{i \in [n] \mid x_i = 2\} \), \( C_x = \{i \in [n] \mid x_i = 3\} \).

It is easy to see that the function is a bijection since we may define the inverse of this function \( e : \{(A, B, C) \mid A, B, C \subseteq [n] \text{ and } A, B, C \text{ are pairwise disjoint}\} \to \{0, 1, 2, 3\}^n \) such that \( e(A, B, C) = (x_1, \ldots, x_n) \), where \( x_i = \begin{cases} 1 & \text{if } i \in A \\ 2 & \text{if } i \in B \\ 3 & \text{if } i \in C \\ 0 & \text{otherwise} \end{cases} \).

- Let \( f(e(A, B, C)) = (A', B', C') \) and \( e(A, B, C) = (x_1, \ldots, x_n) \). Note that \( x_i = 1 \) iff \( i \in A \) and \( i \in A' \) iff \( x_i = 1 \); hence \( i \in A \) iff \( i \in A' \). In other words, \( A = A' \). Similarly we may consider other cases (we use the fact that \( A, B, \) and \( C \) to show that constraints in the definition of \( e \) cannot be satisfied simultaneously).

- Let \( e(f(x_1, \ldots, x_n)) = (x'_1, \ldots, x'_n) \) and \( f(x_1, \ldots, x_n) = (A, B, C) \). Note that \( i \in A \) iff \( x_i = 1 \) and \( x'_i = 1 \) iff \( i \in A \); hence \( x_i = 1 \) iff \( x'_i = 1 \). Similarly we may prove for 0, 2, and 3 and as a result, we proved that \( x_i = x'_i \).
4. (10 points) How many numbers from \([999]\) are not divisible neither by 3, nor by 5, nor by 7.

**Solution:** Let \(S_n = \{i \in [999] \mid \text{i is divisible by n}\}\). Note that \(S_3 \cap S_5 = S_{15}\), \(S_3 \cap S_7 = S_{21}\), \(S_5 \cap S_7 = S_{35}\), and finally, \(S_3 \cap S_5 \cap S_7 = S_{105}\). Additionally, \(|S_3| = \frac{999}{3} = 333\), \(|S_5| = \frac{999}{5} = 199\), \(|S_7| = \frac{999}{7} = 142\), \(|S_{15}| = \frac{999}{15} = 66\), \(|S_{21}| = \frac{999}{21} = 47\), \(|S_{35}| = \frac{999}{35} = 28\), and \(|S_{105}| = \frac{999}{105} = 9\). As a result, by the inclusion-exclusion principle, the answer is \(999 - 333 - 199 - 142 + 66 + 47 + 28 - 9 = 457\).
5. (10 points) Let $m$ be some integer. Show that product of $m$ consecutive integers is divisible by $m!$.

**Solution:** In other words we need to show that for any integer $n$, \( \frac{n(n+1)\cdots(n+m-1)}{m!} \) is an integer. But one may notice that \( \frac{n(n+1)\cdots(n+m-1)}{m!} = \binom{n+m-1}{m} \) which is an integer.