1. Prove that \(1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \leq 2\).

**Solution:** Let us prove a stronger statement:

\[
1 + \frac{1}{2^2} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}.
\]

The base case is clear. We prove now the induction step. By the induction hypothesis the following inequality holds

\[
1 + \frac{1}{2^2} + \cdots + \frac{1}{(n-1)^2} \leq 2 - \frac{1}{n-1}.
\]

Hence,

\[
1 + \frac{1}{2^2} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n-1} + \frac{1}{n^2} = 2
\]

but \(\frac{1}{n-1} - \frac{1}{n^2} \geq \frac{1}{n}\). Indeed, \(\frac{1}{n-1} \geq \frac{n+1}{n^2}\) is equivalent to \(\frac{1}{(n-1)^2} \geq \frac{1}{n^2}\) which is true. As a result we proved the induction step.
2. Show that \((1 + x)^n \geq 1 + nx\) for every \(n \in \mathbb{N}\) and \(x \geq -1\).

**Solution:** We are going to prove that by induction by \(n\).

The induction step is clear. We prove now the induction step. Assume the induction hypothesis: 
\((1 + x)^n \geq 1 + nx\). Note that 
\[(1 + x)^{n+1} \geq (1 + nx) \cdot (1 + x) = 1 + nx + x + nx^2 \geq 1 + (n + 1)x.\]
Which proves the induction step.
3. There are irrational numbers $a$ and $b$ such that $a^b$ is rational.

**Solution:** Assume that $a^b$ is irrational for all irrationals $a$ and $b$. Since $\sqrt{2}$ is irrational $\sqrt{2}^{\sqrt{2}}$ is also irrational. However, $(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2^2} = 2$, which contradicts the assumption.

As a result, there are irrational $a$ and $b$ such that $a^b$ is rational.
4. If $a, b \in \mathbb{Z}$, then $a^2 - 4b + 2 \neq 0$.

**Solution:** Assume, for the sake of contradiction, that such $a$ and $b$ exist. Note that $a^2 = 4b + 2$, hence, $a$ is even but $a$ is not divisible for 4 which is a contradiction.