Show all of your work. Full credit will be given only for answers with explanations.

1. (50 points) Check all the correct statements.

- The tangent plane of the function \( f(x, y) = xe^y + ye^x \) at \((1, 1, 2e)\) is defined by the equation \(2ex + 2ey = z + 2e\).
- The angle between \(\frac{\partial f}{\partial x}(\pi, 0)\) and \(\frac{\partial f}{\partial y}(\pi, 0)\) is \(\pi/2\), where \(f(x, y) = \langle \cos(x) + \sin(y), \sin(x) + \cos(y) \rangle\).
- If \(z = x^2 + y^2\), \(x = \sin(t)\), and \(y = \cos(t)\), then \(\frac{dz}{dt} = 0\).
- The tangent planes of \(f(x, y) = x^2 + y^2\) at \((1, 0, 1)\) and \((0, 1, 1)\) are parallel.
- The vector \(\langle 1, -1, 1 \rangle\) is perpendicular to \(\frac{df(x)}{dt}\) and \(\frac{df(\pi/2)}{dt}\), where \(f(t) = \langle \cos(t), \sin(t), t \rangle\).

Solution:

- First of all we need to compute the partial derivatives of \(f\), \(\frac{\partial f}{\partial x} = e^y + ye^x\) and \(\frac{\partial f}{\partial y} = xe^y + e^x\).
  If \(x = 1\) and \(y = 1\), then \(\frac{\partial f}{\partial x} = 2e\) and \(\frac{\partial f}{\partial y} = 2e\). As a result the tangent plane is \(2e(x-1) + 2e(y-1) = z - 2e\) which can be simplified to \(2ex + 2ey = z + 2e\).
- Let us first find explicitly the derivatives, \(\frac{df}{dx} = \langle -\sin(x), \cos(x) \rangle\) and \(\frac{df}{dy} = \langle \cos(y), -\sin(x) \rangle\).
  At point \((\pi, 0)\) the value of the derivatives are equal to \(\langle 0, -1 \rangle\) and \(\langle 1, 0 \rangle\). The angle between these vectors is \(\frac{\pi}{2}\).
- Note that \(x^2 + y^2 = \cos^2(t) + \sin^2(t)\) which is equal to 1 for any \(t\). Hence, \(z = 1\) and \(\frac{dz}{dt} = 0\).
- We need to find the partial derivatives of \(f\), \(\frac{df}{dx} = 2x\) and \(\frac{df}{dy} = 2y\). Hence, the tangent planes for this surface are \(z = 2(x-1)\) and \(z = 2(y-1)\) and they are not parallel.
- Let us find the derivative if \(f\), \(\frac{df}{dt} = \langle -\sin(t), \cos(t), 1 \rangle\). Hence, \(\frac{df(\pi)}{dt} = \langle 0, -1, 1 \rangle\) and \(\frac{df(\pi/2)}{dt} = \langle -1, 0, 1 \rangle\). Since \(\langle 1, -1, 1 \rangle \cdot \langle 0, -1, 1 \rangle = 2\) they are not perpendicular.
2. Let \( r = (x^y, y^x) \), where \( x = e^t \) and \( y = t^2 \).

(a) (5 points) Find \( \frac{dr}{dt} \).

**Solution:** First of all, we need to find \( \frac{dx^y}{dt} \) and \( \frac{dy^x}{dt} \).

\[
\frac{dx^y}{dt} = \frac{\partial x^y}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial x^y}{\partial y} \cdot \frac{dy}{dt} = (yx^{y-1}) \cdot e^t + (x^y \ln x) \cdot 2t
\]

\[
\frac{dy^x}{dt} = \frac{\partial y^x}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial y^x}{\partial y} \cdot \frac{dy}{dt} = (y^x \ln y) \cdot e^t + (xy^{x-1}) \cdot 2t
\]

As a result,

\[
\frac{dr}{dt} = [(yx^{y-1}) \cdot e^t + (x^y \ln x) \cdot 2t, (y^x \ln y) \cdot e^t + (xy^{x-1}) \cdot 2t].
\]

(b) (5 points) Find the tangent line of the curve described by the vector function \( r \) for \( t = 1 \)

**Solution:** If \( t = 1 \), then \( x = e \), \( y = 1 \), and \( r = (e, 1) \). Hence, \( \frac{dr}{dt} = (1 \cdot 1 \cdot e^1 + (e \cdot 1) \cdot 2, 1 \cdot 0 \cdot e + (1 \cdot e) \cdot 2) = (3e, 2e) \).

As a result, the answer is the line going through \( (e, 1) \) with the slope \( \frac{2e}{3e} \). Hence, the answer is

\[
y = \frac{2}{3}(x - e) + 1.
\]
3. Let \( f(x, y) = xy^2 + yx^2 \).

(a) (5 points) Find the tangent planes to the surface defined by \( f \) at \((1, 1, 2)\) and \((-1, -1, -2)\).

Solution:

(b) (5 points) Check if these planes are intersecting; if they are intersecting, find symmetric equations for the line of intersection of the planes.

Solution:
Let us consider a surface defined implicitly by the equation \( x^3 + y^3 + z^3 + 6xyz = 1 \). Find the tangent plane of the surface at \((1, -3, -3)\).

**Solution:** Let us compute the partial derivative by \( x \) of both sides of the equality

\[
x^3 + y^3 + z^3 + 6xyz = 1
\]

we get \( 3x^2 + 3z^2 \frac{\partial z}{\partial x} + 6yz + 6xy \frac{\partial z}{\partial x} = 0 \). As a result, \( \frac{\partial z}{\partial x} = -\frac{3x^2 + 6yz}{3z^2 + 6xy} \). Similarly, \( \frac{\partial z}{\partial y} = -\frac{3y^2 + 6xz}{3z^2 + 6xy} \).

Since \( x = 1 \), \( y = -3 \), and \( z = -3 \), we may find the values of the partial derivatives: \( \frac{\partial z}{\partial x} = -\frac{3 + 6 \cdot 9}{27 - 6 \cdot 3} = -\frac{57}{9} \) and \( \frac{\partial z}{\partial y} = -\frac{27 - 6 \cdot 3}{27 - 6 \cdot 3} = -1 \).

Therefore, the tangent plane to the surface is defined by the equation \( z + 3 = -\frac{57}{9}(x - 1) - (x + 3) \).