1. (10 points) In the subtraction game with two piles where players may subtract 1, 2 or 5 chips on their turn, identify the N- and P-positions. (Please do not forget to prove correctness of your answer.)

Solution: Let us prove that

\[
g(x) = \begin{cases} 
0 & \text{if } x \equiv 0 \pmod{3} \\
1 & \text{if } x \equiv 1 \pmod{3} \\
2 & \text{if } x \equiv 2 \pmod{3} 
\end{cases}
\]

is the Sprague–Grundy function for the subtraction game with one pile.

We prove the statement using induction by \(x\). First we need to prove the base cases for \(x \leq 4\).

- It is clear that \(g(0) = 0\) since 0 is a terminal position.
- From 1 there is only one move to 0; hence, \(g(1) = \operatorname{mex}\{0\} = 1\).
- From 2 there are two moves to 0 and to 1; hence, \(g(2) = \operatorname{mex}\{0, 1\} = 2\).
- From 3 there are two moves to 1 and to 2; hence, \(g(3) = \operatorname{mex}\{1, 2\} = 0\).
- From 4 there are two moves to 0 and to 1; hence, \(g(4) = \operatorname{mex}\{2, 0\} = 1\).

Assume that the statement is true for all \(y < x\). Note that there are three moves from \(x\): to \(x - 1\), to \(x - 2\), and to \(x - 5\). It is easy to see that \(x - 5 \equiv x - 2 \pmod{3}\); hence \(g(x) = \operatorname{mex}\{g(x - 1), g(x - 2)\}\). Therefore, by considering three cases of the reminder of \(x\) modulo 3 we can prove the statement.
2. (10 points) Alice and Bob have several piles of chips. On each turn they can either remove 1 or 2 chips from one pile, or split a pile into two nonempty piles. Players take turns and a player that cannot make a move loses. Find the value of the Sprague–Grundy function for positions with one pile made of \(n\) chips. (Please do not forget to prove correctness of your answer.)

**Solution:** Let \(g\) be the Sprague–Grundy function for this game. It is clear that the position \((x, y)\) (the position with two piles having \(x\) and \(y\) chips, respectively) is equivalent to the position \((x, y)\) in the same of this game with itself. Hence, \(g(x, y) = g(x) \oplus g(y)\).

As a result, 

\[
g(x) = \text{mex}(\{g(x-1), g(x-2)\} \cup \{g(y) \oplus g(z) : y, z \geq 1, y + z = x\})
\]

for \(x \geq 2\) and \(g(0) = 1\) and \(g(1) = 1\).

Let us prove that 

\[
g(x) = \begin{cases} 
0 & \text{if } x = 0 \\
1 & \text{if } x \equiv 1 \pmod{4} \\
2 & \text{if } x \equiv 2 \pmod{4} \\
0 & \text{if } x \equiv 3 \pmod{4} \\
3 & \text{if } x \equiv 0 \pmod{4}
\end{cases}
\]

The base case for \(x \leq 4\) is clear.

- By the above formula, \(g(2) = \text{mex}\{g(0), g(1), g(1) \oplus g(1)\} = \text{mex}\{0, 1\} = 2\).
- By the above formula, \(g(3) = \text{mex}\{g(1), g(2), g(1) \oplus g(2)\} = \text{mex}\{1, 2, 3\} = 0\).
- By the above formula, \(g(4) = \text{mex}\{g(2), g(3), g(1) \oplus g(3), g(2) \oplus g(2)\} = \text{mex}\{2, 0, 1, 0\} = 3\).

Let us now prove the induction step. Assume the statement is true for all \(y < x\).

- Let \(x \equiv 1 \pmod{4}\) Assume \(y + z = x\) and \(y, z \geq 1\). We calim that \(g(y) \oplus g(z) \neq 1\). Indeed, the only pairs of numbers whose xor gives 1 among 0, 1, 2, 3 are 0 and 1, and 2 and 3.
  - If \(g(y) = 0\) and \(g(z) = 1\), then \(y \equiv 3 \pmod{4}\) and \(z \equiv 1 \pmod{4}\). Which implies that \(y + z \equiv 0 \pmod{4}\) and this contradicts to the assumption.
  - If \(g(y) = 2\) and \(g(z) = 3\), then \(y \equiv 2 \pmod{4}\) and \(z \equiv 0 \pmod{4}\). Which implies that \(y + z \equiv 2 \pmod{4}\) and this contradicts to the assumption.

  However, \(g(x-1) = 3\) and \(g(x-2) = 0\). Hence, \(g(x) = 1\).

- Let \(x \equiv 2 \pmod{4}\) Assume \(y + z = x\) and \(y, z \geq 1\). We calim that \(g(y) \oplus g(z) \neq 2\). Indeed, the only pairs of numbers whose xor gives 2 among 0, 1, 2, 3 are 0 and 2, and 1 and 3.
  - If \(g(y) = 0\) and \(g(z) = 2\), then \(y \equiv 3 \pmod{4}\) and \(z \equiv 2 \pmod{4}\). Which implies that \(y + z \equiv 1 \pmod{4}\) and this contradicts to the assumption.
  - If \(g(y) = 1\) and \(g(z) = 3\), then \(y \equiv 1 \pmod{4}\) and \(z \equiv 0 \pmod{4}\). Which implies that \(y + z \equiv 1 \pmod{4}\) and this contradicts to the assumption.

  However, \(g(x-1) = 1\), \(g(x-2) = 3\), and \(g(x-1) \oplus g(1) = 0\). Hence, \(g(x) = 2\).

- Let \(x \equiv 3 \pmod{4}\) Assume \(y + z = x\) and \(y, z \geq 1\). We calim that \(g(y) \oplus g(z) \neq 0\). Indeed, the only pairs of numbers whose xor gives 3 among 0, 1, 2, 3 are the equal pairs
– If \( g(y) = 0 \) and \( g(z) = 0 \), then \( y \equiv 3 \pmod{4} \) and \( z \equiv 3 \pmod{4} \). Which implies that \( y + z \equiv 2 \pmod{4} \) and this contradicts to the assumption.

– If \( g(y) = 1 \) and \( g(z) = 1 \), then \( y \equiv 1 \pmod{4} \) and \( z \equiv 1 \pmod{4} \). Which implies that \( y + z \equiv 2 \pmod{4} \) and this contradicts to the assumption.

– If \( g(y) = 2 \) and \( g(z) = 2 \), then \( y \equiv 2 \pmod{4} \) and \( z \equiv 2 \pmod{4} \). Which implies that \( y + z \equiv 0 \pmod{4} \) and this contradicts to the assumption.

– If \( g(y) = 3 \) and \( g(z) = 3 \), then \( y \equiv 0 \pmod{4} \) and \( z \equiv 0 \pmod{4} \). Which implies that \( y + z \equiv 0 \pmod{4} \) and this contradicts to the assumption.

However, \( g(x - 1) = 2, g(x - 2) = 1 \).

• Let \( x \equiv 0 \pmod{4} \) Assume \( y + z = x \) and \( y, z \geq 1 \). We claim that \( g(y) \oplus g(z) \neq 3 \). Indeed, xor of two numbers among 0, 1, 2, 3 is at most 3. However, \( g(x - 1) = 0, g(x - 2) = 2, \) and \( g(x - 1) \oplus g(1) = 1 \). Hence, \( g(x) = 3 \).