1. (10 points) Let $a_n$ be a sequence such that $a_1 = 9$, $a_2 = 41$, and $a_{n+2} = 9a_{n+1} - 20a_n$. Show that $a_n = 4^n + 5^n$.

**Solution:** We prove the statement using induction by $n$. The base cases for $n = 1$ and $n = 2$ are true since $a_1 = 9 = 4^1 + 5^1$ and $a_2 = 41 = 4^2 + 5^2$.

Let us prove the induction step from $n$ to $n + 1$. The induction hypothesis is that $a_k = 4^k + 5^k$ for all $k \leq n$. Note that

$$a_{n+1} = 9a_n - 20a_{n-1} = 9(4^n + 5^n) - 20(4^{n-1} + 5^{n-1}) =$$

$$36 \cdot 4^{n-1} + 45 \cdot 5^{n-1} - 20 \cdot 4^{n-1} + 20 \cdot 5^{n-1} =$$

$$16 \cdot 4^{n-1} 25 \cdot 5^{n-1} = 4^{n+1} + 5^{n+1}.$$
2. We say that \( L \) is a list of powers of \( x \) iff
   - either \( L = x^k \) for some positive integer \( k \) or
   - \( L = (x^k, L') \) where \( L' \) is a list of powers of \( x \) and \( k \) is a positive integer.

Let \( L \) be a list of powers of \( x \). We say that the sum of \( L \) with \( x = v \) denoted by \( \sum L \mid_{x=v} \)
   - is equal to \( v^k \) whether \( L = x^k \) and
   - is equal to \( v^k + \sum L' \mid_{x=v} \) whether \( L = (x^k, L') \).

Prove that for any list \( L \) of powers of \( x \) there is a polynomial such that \( \sum L \mid_{x=v} = p(v) \) for all real numbers \( v \).

**Solution:** We prove the statement using the structural induction. The base case is when the list \( L \)
is equal to \( x^k \). Consider the polynomial \( p(x) = x^k \) and note that \( \sum L \mid_{x=v} = v^k = p(v) \). Hence, the
base case is true.

Now we prove the induction step. Consider \( L = (x^k, L') \), by the induction hypothesis there is a
polynomial \( q(x) \) such that \( \sum L' \mid_{x=v} = q(v) \) for any real \( v \). Let us define the polynomial \( p(x) = x^k + q(x) \). It is easy to see that \( \sum L \mid_{x=v} = v^k + \sum L' \mid_{x=v} = v^k + q(v) = p(v) \).
3. (10 points) Prove that $\sum_{i=1}^{n} (i+1)2^i = n2^{n+1}$ for all integers $n \geq 1$.

**Solution:** First we prove the base case for $n = 1$. Note that $\sum_{i=1}^{1} (i+1)2^i = 2 \cdot 2 = 2^2$; hence, the base case is true. Let us check the induction step from $k$ to $k+1$. By the induction hypothesis $\sum_{i=1}^{k} (i+1)2^i = k2^{k+1}$ It is clear that

$$\sum_{i=1}^{k+1} (i+1)2^i = \sum_{i=1}^{k} (i+1)2^i + (k+2)2^{k+1} = k2^{k+1} + (k+2)2^{k+1} = 2(k+1)2^{k+1} = (k+1)2^{k+2}.$$