1. (10 points) Let $a_n$ be a sequence such that $a_1 = 9$, $a_2 = 41$, and $a_{n+2} = 9a_{n+1} - 20a_n$. Show that $a_n = 4^n + 5^n$.

**Solution:** We prove the statement using induction by $n$. The base cases for $n = 1$ and $n = 2$ are true since $a_1 = 9 = 4^1 + 5^1$ and $a_2 = 41 = 4^2 + 5^2$.

Let us prove the induction step from $n$ to $n+1$. The induction hypothesis is that $a_k = 4^k + 5^k$ for all $k \leq n$. Note that

$$a_{n+1} = 9a_n - 20a_{n-1} = 9(4^n + 5^n) - 20(4^{n-1} + 5^{n-1}) =$$

$$= 36 \cdot 4^{n-1} + 45 \cdot 5^{n-1} - 20 \cdot 4^{n-1} + 20 \cdot 5^{n-1} =$$

$$= 16 \cdot 4^{n-1} + 25 \cdot 5^{n-1} = 4^{n+1} + 5^{n+1}.$$
2. We say that $L$ is a list of powers of $x$ iff
   
   - either $L = x^k$ for some positive integer $k$ or
   - $L = (x^k, L')$ where $L'$ is a list of powers of $x$ and $k$ is a positive integer.

Let $L$ be a list of powers of $x$. We say that the sum of $L$ with $x = v$ denoted by $\sum L\big|_{x=v}$
   - is equal to $v^k$ whether $L = x^k$ and
   - is equal to $v^k + \sum L'\big|_{x=v}$ whether $L = (x^k, L')$.

Prove that for any list $L$ of powers of $x$ there is a polynomial such that $\sum L\big|_{x=v} = p(v)$ for all real numbers $v$.

**Solution:** We prove the statement using the structural induction. The base case is when the list $L$ is equal to $x^k$. Consider the polynomial $p(x) = x^k$ and note that $\sum L\big|_{x=v} = v^k = p(v)$. Hence, the base case is true.

Now we prove the induction step. Consider $L = (x^k, L')$, by the induction hypothesis there is a polynomial $q(x)$ such that $\sum L'\big|_{x=v} = q(v)$ for any real $v$. Let us define the polynomial $p(x) = x^k + q(x)$. It is easy to see that $\sum L\big|_{x=v} = v^k + \sum L'\big|_{x=v} = v^k + q(v) = p(v)$. 
3. (10 points) Prove that \( \sum_{i=1}^{n} (i + 1)2^i = n2^{n+1} \) for all integers \( n \geq 1 \).

**Solution:** First we prove the base case for \( n = 1 \). Note that \( \sum_{i=1}^{1} (i + 1)2^i = 2 \cdot 2 = 2^2 \); hence, the base case is true. Let us check the induction step from \( k \) to \( k + 1 \). By the induction hypothesis \( \sum_{i=1}^{k} (i + 1)2^i = k2^{k+1} \) It is clear that

\[
\sum_{i=1}^{k+1} (i + 1)2^i = \sum_{i=1}^{k} (i + 1)2^i + (k + 2)2^{k+1} = k2^{k+1} + (k + 2)2^{k+1} = 2(k + 1)2^{k+1} = (k + 1)2^{k+2}.
\]