1. (10 points) Let us define a union of more than two sets as follows. Let $A_1, \ldots, A_n$ be some sets. Then

- $\bigcup_{i=1}^1 A_i = A_1$ and
- $\bigcup_{i=1}^{k+1} A_i = \left( \bigcup_{i=1}^k A_i \right) \cup A_{k+1}$.

Show that $\bigcup_{i=1}^n [i] = [n]$ for all integers $n > 0$.

**Solution:** First, note that $[k-1] \cup [k] = [k]$ for any $k > 1$.

Now we are ready to prove the statement, we prove it using induction by $n$. The base case is true since $\bigcup_{i=1}^1 [i] = [1]$ by the definition. The induction step is also true since by the induction hypothesis $\bigcup_{i=1}^{k-1} [i] = [k-1]$ and by the definition of the union $\bigcup_{i=1}^k [i] = \bigcup_{i=1}^{k-1} [i] \cup [k]$. Hence, $\bigcup_{i=1}^k [i] = \bigcup_{i=1}^{k-1} [i] \cup [k] = [k-1] \cup [k] = [k]$. 

2. (10 points) Let us define an intersection of more than two sets as follows. Let $A_1, \ldots, A_n$ be some sets. Then

- $\bigcap_{i=1}^1 A_i = A_1$ and
- $\bigcap_{i=1}^{k+1} A_i = \left( \bigcap_{i=1}^k A_i \right) \cap A_{k+1}$.

Show that $\bigcap_{i=1}^n \{ x \in \mathbb{N} : i \leq x \leq n \} = \{ n \}$ for all integers $n > 0$.

**Solution:** There are two solutions for this problem which provide two important points of view.

1. We may prove the statement using induction by $k$. Note that the statement does not involve $k$ so we need to prove a bit different statement, we prove $\bigcap_{i=1}^k \{ x \in \mathbb{N} : i \leq x \leq n \} = \{ k, k + 1, \ldots, n \}$. The base case for $k = 1$ is clear since $\bigcap_{i=1}^1 \{ x \in \mathbb{N} : i \leq x \leq n \} = \{ x \in \mathbb{N} : 1 \leq x \leq n \} = \{ 1, 2, \ldots, n \}$.

   The induction step is also easy to see. By the induction hypothesis $\bigcap_{i=1}^k \{ x \in \mathbb{N} : i \leq x \leq n \} = \{ k, k + 1, \ldots, n \}$. Note that $k + 1 \bigcap_{i=1}^i \{ x \in \mathbb{N} : i \leq x \leq n \} = \{ k, k + 1, \ldots, n \} \cap \{ k + 1, \ldots, n \} = \{ k + 1, \ldots, n \}$. Thus for $k = n$, $\bigcap_{i=1}^n \{ x \in \mathbb{N} : i \leq x \leq n \} = \{ n \}$.

2. Another approach is to prove the statement using induction by $n$. The base case is also clear, $\bigcap_{i=1}^1 \{ x \in \mathbb{N} : i \leq x \leq 1 \} = \{ 1 \}$.

   Let us now prove the induction step. By the induction hypothesis $\bigcap_{i=1}^k \{ x \in \mathbb{N} : i \leq x \leq k \} = \{ k \}$. Hence,

   $\bigcap_{i=1}^k \{ x \in \mathbb{N} : i \leq x \leq k + 1 \} = \bigcap_{i=1}^k \{ x \in \mathbb{N} : i \leq x \leq k \} \cup \{ k + 1 \} \ast$  

   $\bigcap_{i=1}^k \{ x \in \mathbb{N} : i \leq x \leq k \} \cup \{ k + 1 \} = \{ k, k + 1 \}$; note that all these equalities are clear except the equality marked by $\ast$, we prove it later.
Thus by the definition of the union
\[
\bigcap_{i=1}^{k+1} \{ x \in \mathbb{N} : i \leq x \leq k+1 \} =
\bigcap_{i=1}^{k} \{ x \in \mathbb{N} : i \leq x \leq k+1 \} \cap \{ x \in \mathbb{N} : k+1 \leq x \leq k+1 \} =
\bigcap_{i=1}^{k} \{ x \in \mathbb{N} : i \leq x \leq k+1 \} \cap \{ k+1 \} =
\{ k, k+1 \} \cap \{ k+1 \} = \{ k+1 \}.
\]

Now we need to prove that if \( A_1, \ldots, A_\ell, \) and \( B \) are some sets, then \( \bigcap_{i=1}^{\ell} (A_i \cup B) = \left( \bigcap_{i=1}^{\ell} A_i \right) \cup B. \) We prove it also using induction by \( \ell. \) For \( \ell = 1, \) the statement is clear since \( \bigcap_{i=1}^{1} (A_i \cup B) = A_1 \cup B = \left( \bigcap_{i=1}^{1} A_i \right) \cup B. \) Now we need to check the induction step. By the induction hypothesis \( \bigcap_{i=1}^{k} (A_i \cup B) = \left( \bigcap_{i=1}^{k} A_i \right) \cup B. \) Note that
\[
\bigcap_{i=1}^{k+1} (A_i \cup B) = \bigcap_{i=1}^{k} (A_i \cup B) \cap (A_{k+1} \cup B) = \left( \left( \bigcap_{i=1}^{k} A_i \right) \cup B \right) \cap (A_{k+1} \cup B) =
\left( \left( \bigcap_{i=1}^{k} A_i \right) \cap A_{k+1} \right) \cup B = \left( \bigcap_{i=1}^{k+1} A_i \right) \cup B.
\]