1. (10 points) Let us define a union of more than two sets as follows. Let $A_1, \ldots, A_n$ be some sets. Then

- $\bigcup_{i=1}^1 A_i = A_1$ and
- $\bigcup_{i=1}^{k+1} A_i = \left( \bigcup_{i=1}^k A_i \right) \cup A_{k+1}$.

Show that $\bigcup_{i=1}^n [i] = [n]$ for all integers $n > 0$.

**Solution:** First, note that $[k - 1] \cup [k] = [k]$ for any $k > 1$.

Now we are ready to prove the statement, we prove it using induction by $n$. The base case is true since $\bigcup_{i=1}^1 [i] = [1]$ by the definition. The induction step is also true since by the induction hypothesis $\bigcup_{i=1}^{k-1} [i] = [k - 1]$ and by the definition of the union $\bigcup_{i=1}^k [i] = \bigcup_{i=1}^{k-1} [i] \cup [k]$. Hence, $\bigcup_{i=1}^k [i] = \bigcup_{i=1}^{k-1} [i] \cup [k] = [k - 1] \cup [k] = [k]$. 
2. (10 points) Let us define an intersection of more than two sets as follows. Let \( A_1, \ldots, A_n \) be some sets. Then

- \( \bigcap_{i=1}^{1} A_i = A_1 \) and
- \( \bigcap_{i=1}^{k+1} A_i = \left( \bigcap_{i=1}^{k} A_i \right) \cap A_{k+1} \).

Show that \( \bigcap_{i=1}^{n} \{x \in \mathbb{N} : i \leq x \leq n\} = \{n\} \) for all integers \( n > 0 \).

**Solution:** There are two solutions for this problem which provide two important points of view.

1. We may prove the statement using induction by \( k \). Note that the statement does not involve \( k \) so we need to prove a bit different statement, we prove \( \bigcap_{i=1}^{k} \{x \in \mathbb{N} : i \leq x \leq n\} = \{k, k + 1, \ldots, n\} \). The base case for \( k = 1 \) is clear since \( \bigcap_{i=1}^{1} \{x \in \mathbb{N} : i \leq x \leq n\} = \{x \in \mathbb{N} : 1 \leq x \leq n\} = \{1, 2, \ldots, n\} \).

   The induction step is also easy to see. By the induction hypothesis \( \bigcap_{i=1}^{k} \{x \in \mathbb{N} : i \leq x \leq n\} = \{k, k + 1, \ldots, n\} \). Note that

   \[
   \bigcap_{i=1}^{k+1} \{x \in \mathbb{N} : i \leq x \leq n\} = \bigcap_{i=1}^{k} \{x \in \mathbb{N} : i \leq x \leq n\} \cap \{x \in \mathbb{N} : k + 1 \leq x \leq n\} = \{k, k + 1, \ldots, n\} \cap \{k + 1, \ldots, n\} = \{k + 1, \ldots, n\}.
   \]

   Thus for \( k = n \), \( \bigcap_{i=1}^{n} \{x \in \mathbb{N} : i \leq x \leq n\} = \{n\} \).

2. Another approach is to prove the statement using induction by \( n \). The base case is also clear, \( \bigcap_{i=1}^{1} \{x \in \mathbb{N} : i \leq x \leq 1\} = \{1\} \).

   Let us now prove the induction step. By the induction hypothesis \( \bigcap_{i=1}^{k} \{x \in \mathbb{N} : i \leq x \leq k\} = \{k\} \). Hence,

   \[
   \bigcap_{i=1}^{k} \{x \in \mathbb{N} : i \leq x \leq k + 1\} = \bigcap_{i=1}^{k} \{x \in \mathbb{N} : i \leq x \leq k\} \cup \{k + 1\} = \{k, k + 1\};
   \]

   note that all these equalities are clear except the equality marked by *, we prove it later.
Thus by the definition of the union
\[
\bigcap_{i=1}^{k+1}\{x \in \mathbb{N} : i \leq x \leq k + 1\} = \\
\bigcap_{i=1}^{k}\{x \in \mathbb{N} : i \leq x \leq k + 1\} \cap \{x \in \mathbb{N} : k + 1 \leq x \leq k + 1\} = \\
\bigcap_{i=1}^{k}\{x \in \mathbb{N} : i \leq x \leq k + 1\} \cap \{k + 1\} = \\
\{k, k + 1\} \cap \{k + 1\} = \{k + 1\}.
\]

Now we need to prove that if \(A_1, \ldots, A_\ell,\) and \(B\) are some sets, then \(\bigcap_{i=1}^{\ell}(A_i \cup B) = (\bigcap_{i=1}^{\ell} A_i) \cup B.\) We prove it also using induction by \(\ell.\) For \(\ell = 1,\) the statement is clear since \(\bigcap_{i=1}^{1}(A_i \cup B) = A_1 \cup B = (\bigcap_{i=1}^{1} A_i) \cup B.\) Now we need to check the induction step. By the induction hypothesis
\[
\bigcap_{i=1}^{k}(A_i \cup B) = (\bigcap_{i=1}^{k} A_i) \cup B.
\]
Note that
\[
\bigcap_{i=1}^{k+1}(A_i \cup B) = \bigcap_{i=1}^{k}(A_i \cup B) \cap (A_{k+1} \cup B) = \\
\left(\bigcap_{i=1}^{k} A_i \right) \cup B \cap (A_{k+1} \cup B) = \\
\left(\bigcap_{i=1}^{k} A_i \right) \cap A_{k+1} \cup B = \left(\bigcap_{i=1}^{k+1} A_i \right) \cup B.
\]