1. (10 points) Let \( n \) be a positive integer. Show that in any set of \( n \) consecutive integers there is at least one divisible by \( n \).

**Solution:** Let \( n \) be a positive integer. Show that in any set of \( n \) consecutive integers there is at least one divisible by \( n \).

Let the numbers be \( k, k + 1, \ldots, k + n - 1 \). Assume that all the numbers are not divisible by \( n \). Consider a function \( f : \{k, k + 1, \ldots, k + n - 1\} \rightarrow \{1, \ldots, n - 1\} \) such that \( f(k + i) \) is equal to the reminder of \( k + i \) modulo \( n \). Note that the set on the left has more elements than the set on the right, so there are \( i_1 < i_2 \) such that \( k + i_1 \) has the same reminder as \( k + i_2 \). Thus \( n \) divides \( 0 < i_2 - i_1 < n \), which is a contradiction.
2. Prove that for every integers $a_1, \ldots, a_n$ there are $k > 0$ and $\ell \geq 0$ such that $k + \ell \leq n$ and $\sum_{i=k}^{k+\ell} a_i$ is divisible by $n$.

**Solution:** We consider the $n$ sums modulo $n$ of the form $(a_1)$, $(a_1 + a_2)$, $\ldots$, $(a_1 + a_2 \cdots + a_n)$. First, we note that if any of these sums $a_1 + \cdots + a_\ell \equiv 0 \pmod{n}$, then we are done by picking $k = 1$ and $\ell$ accordingly since being equivalent to 0 (mod n) is the same as being divisible by $n$.

As a result, it suffices to show the result when none of the $n$ sums of the from $(a_1)$, $(a_1 + a_2)$, $\ldots$, $(a_1 + a_2 \cdots + a_n)$ are equivalent to 0 (mod n).

In this case, each of these sums modulo $n$ necessarily must be one of 1, 2, $\ldots$, $(n-1)$. That is, there are $(n-1)$ possible values for each of these $n$ sums. Therefore, by the pigeonhole principle, we must have that two of these sums are equivalent modulo $n$. We thus have that there exists $m > 0$ and $j > 0$ (and without loss of generality may assume that $j > m$ ) so that

$$a_1 + a_2 + \cdots + a_m \equiv a_1 + a_2 + \cdots + a_j \pmod{n}.$$

Now, we note that subtracting $a_1 + a_2 + \cdots + a_m$ from both sides yields that

$$0 \equiv a_{m+1} + \cdots + a_j \pmod{n}$$

and hence by taking $k = m + 1$ and $\ell = j - m - 1 = j - k$ we have that

$$0 \equiv a_k + a_{k+1} + \cdots + a_{k+\ell} \pmod{n}$$

which gives is that $\sum_{i=0}^{\ell} a_{k+i}$ is divisible by $n$ as desired.