1. Show that $\sum_{i=1}^{n} (i+1)2^i = n2^{n+1}$ for all positive integers $n$.

Solution: We prove the statement using induction by $n$. The base case for $n = 1$ is true since $4 = (1 + 1) \cdot 2 = 1 \cdot 2^{1+1} = 4$.

Now we need to prove the induction step. Let us assume that $\sum_{i=1}^{n} (i+1)2^i = n2^{n+1}$. Note that $\sum_{i=1}^{n+1} (i+1)2^i = \sum_{i=1}^{n} (i+1)2^i + (n+2)2^{n+1} = n2^{n+1} + (n+2)2^{n+1} = (2n+2)2^{n+1} = (n+1)2^{n+2}$.

Hence, by the induction principle $\sum_{i=1}^{n} (i+1)2^i = n2^{n+1}$ for any positive integer $n$. 
2. Let $n$ be a positive integer and $A_1, \ldots, A_n$ be some sets. Let us define union of these sets as follows:

1. $\bigcup_{i=1}^{1} A_i = A_1$,
2. $\bigcup_{i=1}^{k+1} A_i = \left( \bigcup_{i=1}^{k} A_i \right) \cup A_{k+1}$.

Show that $\bigcup_{i=1}^{n} [i] = [n]$.

**Solution:** We prove the statement using induction by $n$. If $n = 1$, the statement is true since $\bigcup_{i=1}^{1} [i] = [1] = [1]$.

To prove the induction step we assume that $\bigcup_{i=1}^{n} [i] = [n]$. By the definition of the union $\bigcup_{i=1}^{n+1} [i] = \left( \bigcup_{i=1}^{n} [i] \right) \cup [n+1]$. Hence, $\bigcup_{i=1}^{n+1} [i] = [n] \cup [n+1] = [n+1]$.

Therefore by the induction principle, $\bigcup_{i=1}^{n} [i] = [n]$ for any positive integer $n$. 
3. (10 points) Let $\Omega$ be some set. Consider $A_1, \ldots, A_n \subseteq \Omega$. Show that $\bigcup_{i=1}^n A_i = \{x \in \Omega : \exists i \in [n] \ x \in A_i\}$.

Solution: We prove the statement using induction by $n$. For $n = 1$ the statement is clearly true. Let us now prove the induction step from $n$ to $n+1$. Assume that $\bigcup_{i=1}^n A_i = \{x \in \Omega : \exists i \in [n] \ x \in A_i\}$. Note that $\bigcup_{i=1}^{n+1} A_i = (\bigcup_{i=1}^n A_i) \cup A_{n+1} = \{x \in \Omega : \exists i \in [n] \ x \in A_i\} \cup A_{n+1}$. We denote $\{x \in \Omega : \exists i \in [n] \ x \in A_i\}$ by $B$. By the definition of union, $B \cup A$ is the set of all $x$ such that either $x \in B$ or $x \in A_{n+1}$; therefore

$$B \cup A = \{x \in \Omega : (\exists i \in [n] \ x \in A_i) \text{ or } x \in A_{n+1}\} = \{x \in \Omega : \exists i \in [n+1] \ x \in A_i\}.$$  

Which finishes the proof.