1. (a) (10 points) Let $\phi$, $\psi$, and $\chi$ be propositional formulas on $\Omega$. Show that $(\phi \lor (\psi \land \chi))|_{\rho} = ((\phi \lor \psi) \land (\phi \lor \chi))|_{\rho}$ for any assignment $\rho$ to the variables $\Omega$.

**Solution:** Let us fix some propositional assignment $\rho$ to $\Omega$. Note that by the definition

$$(\phi \lor (\psi \land \chi))|_{\rho} = \phi|_{\rho} \lor (\psi \land \chi)|_{\rho}$$

and

$$(\phi \lor \psi) \land (\phi \lor \chi)|_{\rho} = (\phi|_{\rho} \lor \psi|_{\rho}) \land (\phi|_{\rho} \lor \chi|_{\rho})$$

However, $\phi|_{\rho} \lor (\psi|_{\rho} \land \chi|_{\rho}) = (\phi|_{\rho} \lor \psi|_{\rho}) \land (\phi|_{\rho} \lor \chi|_{\rho})$ by the distributivity of disjunction and conjunction.

(b) (10 points) Let $\psi_{1,1}, \ldots, \psi_{1,n}, \psi_{2,1}, \ldots, \psi_{2,m}$ be propositional formulas on $\Omega$. Let $\phi_1 = \bigwedge_{i=1}^{n} \psi_{1,i}$ and $\phi_2 = \bigwedge_{j=1}^{m} \psi_{2,j}$.

Show that $(\phi_1 \lor \phi_2)|_{\rho} = (\bigwedge_{i=1}^{n} \bigwedge_{j=1}^{m} (\psi_{1,i} \lor \psi_{2,j}))|_{\rho}$ for any assignment $\rho$ to the variables $\Omega$.

**Solution:** Let us again fix some propositional assignment $\rho$ to $\Omega$.

We prove the statement in two steps. In the first one we prove that

$$(\phi_1 \lor \chi)|_{\rho} = \left( \bigwedge_{i=1}^{k+1} \psi_{1,i} \lor \chi \right)|_{\rho}$$

using induction by $n$.

The base case for $n = 1$ is clear. Let us now prove the induction step from $k$ to $k + 1$. Note that

$$\left( \bigwedge_{i=1}^{k+1} \psi_{1,i} \right) \lor \chi = \left( \bigwedge_{i=1}^{k} \psi_{1,i} \right) \lor \psi_{1,k+1} \lor \chi$$

By the previous problem, this implies that

$$\left( \left( \bigwedge_{i=1}^{k+1} \psi_{1,i} \right) \lor \chi \right)|_{\rho} = \left( \left( \bigwedge_{i=1}^{k} \psi_{1,i} \right) \lor \psi_{1,k+1} \lor \chi \right)|_{\rho}$$

The induction hypothesis, implies that

$$\left( \left( \bigwedge_{i=1}^{k+1} \psi_{1,i} \right) \lor \chi \right)|_{\rho} = \left( \left( \bigwedge_{i=1}^{k} \psi_{1,i} \lor \chi \right) \lor \psi_{1,k+1} \lor \chi \right)|_{\rho}$$

Therefore, using the definition of the conjunction of several formulas, we proved that

$$\left( \left( \bigwedge_{i=1}^{k+1} \psi_{1,i} \right) \lor \chi \right)|_{\rho} = \left( \bigwedge_{i=1}^{k+1} \psi_{1,i} \lor \chi \right)|_{\rho}$$

On the second step we prove the statement of the problem using induction by $m$. The base case follows from the result we just proved. Let us now prove the induction step from $k$ to $k + 1$. Note that

$$\left( \bigwedge_{i=1}^{m} \psi_{1,i} \right) \lor \left( \bigwedge_{i=1}^{k+1} \psi_{2,i} \right) = \left( \bigwedge_{i=1}^{m} \psi_{1,i} \right) \lor \left( \left( \bigwedge_{i=1}^{k} \psi_{2,i} \right) \lor \psi_{2,k+1} \right)$$
(c) (10 points) Let $\Omega$ be a set of variables. We say that a propositional formula is a literal if the formula is equal to $x$ or $\neg x$ for $x \in \Omega$.

We say that a propositional formula on $\Omega$ is in conjunctive normal form if it is equal to

$$
\bigwedge_{i=1}^n \psi_{i,j}
$$

be a propositional formula on $\Omega$. Show using structural induction that there is a propositional formula $\psi$ on $\Omega$ in conjunctive normal form such that

$$
\psi = (\bigwedge_{i=1}^m \psi_{1,i}) \wedge (\bigwedge_{i=1}^n \psi_{2,i})
$$

Therefore by the previous statement,

$$
\left( \bigwedge_{i=1}^m \psi_{1,i} \vee \bigwedge_{i=1}^{k+1} \psi_{2,i} \right) = \left( \bigwedge_{i=1}^m (\psi_{1,i} \vee \psi_{2,k+1}) \right) \wedge \left( \bigwedge_{i=1}^{k+1} \psi_{2,i} \right)
$$

Finally, using the induction hypothesis,

$$
\left( \bigwedge_{i=1}^m \psi_{1,i} \vee \bigwedge_{i=1}^{k+1} \psi_{2,i} \right) = \left( \bigwedge_{i=1}^m (\psi_{1,i} \vee \psi_{2,k+1}) \right) \wedge \left( \bigwedge_{i=1}^{k+1} (\psi_{1,i} \vee \psi_{2,i}) \right)
$$

Solution: Before we prove the statement of the problem, we need to show several equalities. Let $\chi_{1,1}, \chi_{2,1}, \ldots, \chi_{1,n_1}, \chi_{2,1}, \ldots, \chi_{2,n_2}, \lambda_1, \ldots, \lambda_{n_1+n_2}$ be propositional formulas over the variables $\Omega$ such that $\chi_{i,j}$ is a literal.

We can prove the statement using induction by $n_2$. The base case for $n_2 = 1$ follows from the definition of the long conjunction. Let us prove the induction step from $k$ to $k + 1$. By the definition of the long conjunction,

$$
\left( \bigwedge_{i=1}^{n_1} \chi_{1,i} \right) \wedge \left( \bigwedge_{i=1}^{n_2} \lambda_{1,i} \right) = \left( \bigwedge_{i=1}^{n_1} \chi_{1,i} \right) \wedge \left( \bigwedge_{i=1}^{k+1} \lambda_{2,i} \wedge \lambda_{2,k+1} \right)
$$

Note that we proved in class the following

$$
\left( \bigwedge_{i=1}^{n_1} \chi_{1,i} \right) \wedge \left( \bigwedge_{i=1}^{n_2} \lambda_{2,i} \right) = \left( \bigwedge_{i=1}^{n_1} \chi_{1,i} \right) \wedge \left( \bigwedge_{i=1}^{k} \lambda_{2,i} \wedge \lambda_{2,k+1} \right)
$$

By the induction hypothesis,

$$
\left( \bigwedge_{i=1}^{n_1} \chi_{1,i} \right) \wedge \left( \bigwedge_{i=1}^{k+1} \lambda_{2,i} \right) = \left( \bigwedge_{i=1}^{n_1+k} \chi_{1,i} \wedge \lambda_{2,k+1} \right)
$$
Which implies the statement by the definition of the long conjunction. Consider $e_{n,m} : \{0, \ldots, m^n - 1\} \rightarrow \{0, \ldots, m - 1\}^n$ be a bijection such that

- $e_{n,m}(i + m \cdot r, 0) = i$
- $e_{n,m}(i + m \cdot r, j) = e_{n-1,m}(r, j - 1),$

for any $0 \leq i < m,$ $0 \leq r < m^{n-1},$ and $0 \leq j < n.$ We also show that $\bigvee_{j=0}^{n-1} \bigwedge_{i=0}^{m-1} \chi_{j,i} = \bigwedge_{q=0}^{m^{n-1} - 1} \bigvee_{j=0}^{n-1} \chi_{j,e_{n,m}(q,j)}.$ We prove the statement using induction by $n.$ The case of $n = 1$ is clear. We prove now the induction step from $k$ to $k + 1.$ Note that

$$\bigvee_{j=0}^{k} \bigwedge_{i=0}^{m-1} X_{j,i} = \left( \bigwedge_{j=0}^{k-1} \bigvee_{i=0}^{m-1} \chi_{j,i} \right) \bigvee \bigwedge_{i=0}^{m-1} X_{k,i}.$$  

By the induction hypothesis,

$$\left( \bigvee_{j=0}^{k} \bigwedge_{i=0}^{m-1} X_{j,i} \right) \big|_{\rho} = \left( \bigwedge_{q=0}^{m^k - 1} \left( \bigvee_{j=0}^{k-1} \chi_{j,e_{n,m}(q,j)} \right) \bigvee \bigwedge_{i=0}^{m-1} \chi_{k,i} \right) \big|_{\rho}.$$  

Therefore,

$$\left( \bigvee_{j=0}^{k} \bigwedge_{i=0}^{m-1} X_{j,i} \right) \big|_{\rho} = \left( \bigwedge_{q=0}^{m^k - 1} \left( \bigvee_{j=0}^{k-1} \chi_{j,e_{n,m}(q,j)} \right) \bigvee \bigwedge_{i=0}^{m-1} X_{k,i} \right) \big|_{\rho}.$$  

So

$$\left( \bigvee_{j=0}^{k} \bigwedge_{i=0}^{m-1} X_{j,i} \right) \big|_{\rho} = \left( \bigwedge_{q=0}^{m^k - 1} \bigwedge_{i=0}^{m-1} \left( \bigvee_{j=0}^{k-1} \chi_{j,e_{n,m}(q,j)} \right) \bigvee \chi_{k,i} \right) \big|_{\rho}.$$  

As a result,

$$\left( \bigvee_{j=0}^{k} \bigwedge_{i=0}^{m-1} X_{j,i} \right) \big|_{\rho} = \left( \bigwedge_{q=0}^{m^{k+1} - 1} \bigwedge_{i=0}^{m-1} \left( \bigvee_{j=0}^{k} \chi_{j,e_{n,m}(q,j)} \right) \right) \big|_{\rho}.$$  

Finally, we are ready to prove the statement of the problem. Let us consider the following cases.

- The first case is when $\phi = x_i.$ In this case we can choose $n = 1,$ $m_1 = 1,$ and $\psi_{1,1} = x_i.$
- The second case is when $\phi = \phi_1 \land \phi_2.$ Note that by the induction hypothesis, $\phi_1 = \bigwedge_{j=1}^{n_1} j_{k=1}^{m_{1,j}} \psi_{1,i,j}$ and $\phi_2 = \bigwedge_{j=1}^{n_2} j_{k=1}^{m_{2,i,j}} \psi_{2,i,j}.$ Using the statement of the previous problem, $\phi = \bigwedge_{j=1}^{n_1+n_2} j_{k=1}^{m_{k,i,j}} \psi_{k,i,j}.$
- The third case is when $\phi = \phi_1 \lor \phi_2.$ Note that by the induction hypothesis, $\phi_1 = \bigwedge_{j=1}^{n_1} j_{k=1}^{m_{1,j}} \psi_{1,i,j}$ and $\phi_2 = \bigwedge_{j=1}^{n_2} j_{k=1}^{m_{2,i,j}} \psi_{2,i,j}.$ Using the just proved statement we may conclude that $\phi = \bigwedge_{j=1}^{n_1+n_2} j_{k=1}^{m_{k,i,j}} \psi_{k,i,j},$ where $m_k = m_{1,i} + m_{2,i} - n_1$ for $n_1 < i \leq n_1 + n_2,$ $\psi_{k,i,j} = \psi_{1,i,j}$ for $1 \leq i \leq n_1$ and $\psi_{k,i,j} = \psi_{2,i,n_1-j}$ for $n_1 < i \leq n_1 + n_2.$
- Finally, the last case is when $\phi = \neg \phi'.$ Note that by the induction hypothesis, $\phi' = \bigwedge_{j=1}^{n_1'} j_{k=1}^{m_{k,i}' \psi_{k, i,j}}.$ Hence, $\phi \big|_{\rho} = \bigwedge_{j=1}^{n_1'} j_{k=1}^{m_{k,i}' \psi_{k, i,j}}.$ And using the second proven observation in this problem we can present a CNF representation of $\phi.$