Lecture 1 Mathematical Induction

Theorem (Induction Principle)
Let \( P(n) \) be some statement about a positive integer \( n \).
\( P(n) \) is true for all positive integers \( n \) iff
- \( P(1) \) is true and
- \( P(k) \) implies \( P(k+1) \).
Example  Show that \( \int_{0}^{\infty} x^n e^{-x} \, dx = n! \)

Proof  Let's start from proving the base case, i.e.

\[ \int_{0}^{\infty} x e^{-x} \, dx = 1 \]

Note that \( \int_{0}^{\infty} x e^{-x} = x (-e^{-x}) \bigg|_{0}^{\infty} - \int_{0}^{\infty} 1 \cdot (-e^{-x}) = \]

\[ = 0 + \int_{0}^{\infty} e^{-x} \, dx = (-e^{-x}) \bigg|_{0}^{\infty} = 1 \]
Now we are ready to prove the induction step from $n$ to $n+1$.

The induction hypothesis is that

$$\int_0^\infty x^n e^{-x} \, dx = n!$$

$$\int_0^\infty x^{n+1} e^{-x} \, dx = x^{n+1} \bigg|_0^\infty - \int_0^\infty (n+1)x^n e^{-x} \, dx =$$

$$= 0 + (n+1) \int_0^\infty x^n e^{-x} \, dx = (n+1)! \uparrow$$

by the I.H.
Exercise: Show that
\[ \sum_{k=1}^{n} k \cdot k! = (n+1)! - 1 \]

The induction step is clear since

\[ 1 \cdot 1! = 2 \cdot 1 - 1. \]

Let us prove the induction step.

By the induction hypothesis, \( \sum_{k=1}^{n} k \cdot k! = (n+1)! - 1 \)

Hence, \( \sum_{k=1}^{n+1} k \cdot k! = \sum_{k=1}^{n} k \cdot k! + (n+1) \cdot (n+1)! = \)

\[ = (n+1)! - 1 + (n+1) \cdot (n+1)! = (n+1)! \cdot (n+1+1)! - 1 = \]

\[ = (n+2)! - 1 \]
Exercise: Show that
\[ \sum_{i=1}^{n} \frac{1}{i^2} \leq 2 \]

It's enough to show that \[ \sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - \frac{1}{n} \]
since \[ 2 - \frac{1}{n} < 2 \].

We prove this using induction by \( n \).

The base case is true since \( \frac{1}{1} \leq 2 - 1 \).

Let us now prove the induction step from \( n \) to \( n+1 \). By the induction hypothesis,
\[ \sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - \frac{1}{n} \].

Note that \[ \sum_{i=1}^{n+1} \frac{1}{i^2} = \]
\[ = \sum_{i=1}^{n} \frac{1}{i^2} + \frac{1}{(n+1)^2} \leq 2 - \frac{1}{n} + \frac{1}{(n+1)^2} = \]
\[ = 2 - \left( \frac{n^2 + 2n + 1 - n}{n(n+1)^2} \right) = 2 - \frac{1}{n+1} \left( \frac{n^2 + n + 1}{n(n+1)} \right) \leq \]
\[ \leq 2 - \frac{1}{n+1} \]