Lecture 5: Structural Induction

\[
(1 \ (2 \ 3))
\]

\[
(1 \ 1)
\]

\[
h((T_1, T_2)) = \max (h(T_1), h(T_2)) + 1
\]

\[
h(i) = 0
\]
Exercise

Define "size" of a binary tree.
Definition \textit{Let }T\text{ be a binary tree}

\textit{(base case)} If \( T \) is an integer, then \( s(T) = 1 \) and \( h(T) = 0 \)

\textit{(recursion step)} Let \( T = (T_1, T_2) \). Then
\[
    s(T) = s(T_1) + s(T_2)
    \quad \text{and}
    \quad h(T) = \max \left( h(T_1), h(T_2) \right) + 1
\]

Exercise

What is the height and size of \( (1 \ (2\ 3)) \) and \( ((1\ 2) \ (1\ 2)) \)
\[
    h((1 \ (2\ 3))) = \max \left\{ h(1), h((2\ 3)) \right\} + 1
\]
\[ h((1 \ (2 \ 3))) = \max \{ h(1), h((2 \ 3)) \} + 1 = \]

\[ = \max \{ 0, \max \{ h(2), h(3) \} \} + 1 = \]

\[ = \max \{ 0, \max \{ 0, 3 \} \} + 1 = \max \{ 0, 1 \} + 1 = 1 + 1 = 2 \]

\[ \sigma((1 \ (2 \ 3))) = \sigma(1) + \sigma((2 \ 3)) = \sigma(1) + \]

\[ - \sigma(2) + \sigma(3) = 3 \]
Let \( U = \mathbb{R} \), \( B = \{0, 1\} \), and \( \mathcal{F} = \{f, g\} \)

Consider \( \nu : S \rightarrow \mathbb{R} \) s.t.

\[
\begin{align*}
- \nu(0) &= 0 \\
- \nu(f(x, y)) &= f(\nu(x), \nu(y)) \\
\nu(g(x)) &= \nu(x) + 1
\end{align*}
\]

Then

\[
\begin{align*}
f(0, 0) &= c \\
\nu(c) &= \nu(f(0, 0)) = f(\nu(0), \nu(0)) \\
\nu(g(0)) &= \nu(0) + 1 = 1 \\
\nu(0) &= 0
\end{align*}
\]
**Definition**

A set $S$ is freely generated by $F$ from $B$ if
- $S$ is generated by $F$ from $B$ and
- $B \cap \text{Im } f, \text{Im } f \cap \text{Im } g = \emptyset$ for any $f, g \in F$

**Theorem** Let $S \subseteq U$ be the set freely generated from $B$ by $F = \{ f_i : U^i \to U \text{ for } 1 \leq i \leq m \}$

Let $F_B : B \to V, F_i : U^i \to V \quad F_n : U^n \to V$

Then there is a function $h : S \to V$ s.t.
- $h(u) = F_B(u)$ for all $u \in B$
- $h(\{ f_i(u, \ldots, u_{e_i}) \}) = F_i(h(u_1), \ldots, h(u_{e_i}))$
Theorem Let \( S \subseteq U \) be the set freely
generated from \( B \) by \( F = \{ F_i : U^n \rightarrow U \} \).

Let \( F_B : B \rightarrow V \), \( F_i : U^n \rightarrow U \)...

Then, there is a function \( h : S \rightarrow V \) s.t
- \( h(u) = F_B(u) \) for all \( u \in B \)
- \( h(f_i(u_1, ..., u_e)) = F_i(h(u_1), ..., h(u_e)) \)

Definition Let \( T \) be a binary tree

(base case) If \( T \) is an integer,
then \( s(T) = 1 \) and \( h(T) = 0 \)

(recursion step) Let \( T = (T_1, T_2) \). Then
\( s(T) = s(T_1) + s(T_2) \) and
\( h(T) = \max (h(T_1), h(T_2)) + 1 \)
Exercise: Define a function that sums up all the numbers in the tree:

\[ s(T) \leq 2 \]

\[ h(T) \]