1. (10 points) Give a natural deduction derivation of $\exists x (A(x) \lor B(x))$ from $\exists x A(x) \lor \exists x B(x)$. 
2. (10 points) Let us consider the following formulas on the variables from the set \( \{x_0, \ldots, x_n\} \).

1. The formula \( I_n \) is equal to \( x_0 \).
2. The formula \( S_{n,i} \) is equal to \( x_{i-1} \Rightarrow x_i \).
3. The formula \( T_n \) is equal to \( x_n \).

Show that there is a natural deduction derivation of \( T_n \) from \( I_n \land \bigwedge_{i=1}^{n} S_{n,i} \).
3. (10 points) Let $\phi = \bigvee_{i=1}^{m} \lambda_i$ be a clause; we say that the width of the clause is equal to $m$. Let $\phi = \bigwedge_{i=1}^{\ell} \chi_i$ be a formula in CNF; we say that the width of $\phi$ is equal to the maximal width of $\chi_i$ for $i \in [\ell]$.

Let $m_n : \{T, F\}^n \rightarrow \{T, F\}$ such that $m_n(x_1, \ldots, x_n) = T$ iff the number of elements in the set $\{i : x_i = T\}$ is divisible by 3.

Show that any CNF representation of $m_n$ has width at least $n - 2$. 
4. (10 points) Let $A \Delta B = (A \cup B) \setminus (A \cap B)$; we say that $A \Delta B$ is the symmetric difference of $A$ and $B$.

Let $\Omega$, and $A_1, \ldots, A_n \subseteq \Omega$ be some sets. We say that $\Delta_{i=1}^n A_i = A_1$ and $\Delta_{i=1}^{k+1} A_i = (\Delta_{i=1}^k A_i) \Delta A_{k+1}$.

Show that

$$\Delta_{i=1}^n A_i = \{x \in \Omega : x \in A_i \text{ for odd number of } i \in [n]\}.$$
5. (10 points) Let $\mathcal{S}$ be a signature with two predicate symbols $=$ and $S$ such that the first is binary and the last is ternary.

Let us consider the structure $\mathfrak{M}$ such that it corresponds to the points on a two-dimensional plane, $=$ is a standard equality, and $S(x, y, z)$ is true iff $|xz| = |yz|$.

Let $R$ be a relation such that $(A, B, C) \in R$ iff $A$, $B$, and $C$ lay on the same line. Show that $R$ is representable in $\mathfrak{M}$.
6. (10 points) Let us define the set $S$ defined as follows:

- $3 \in S$ and
- if $x \in S$ and $y \in S$, then $(x + y) \in S$.

Show that $S = \{3k : k \in \mathbb{N}\}$. 
7. (10 points) Let \( f, g_1, \ldots, g_n : \mathbb{R}^\ell \to \mathbb{R} \) We say that the equation \( f(x) = 0 \) can be derived from the equations \( g_1(x) = 0, \ldots, g_n(x) = 0 \) iff there is a sequence of functions \( h_1, \ldots, h_m : \mathbb{R}^\ell \to \mathbb{R} \) such that
\[ h_m = f \]
and for each \( i \in [m] \),
\[ \begin{align*}
\text{either } h_i & \text{ is equal to } g_j \text{ for some } j \in [n], \text{ or} \\
h_i & = h_j + h_k \text{ for some } 1 \leq j, k < i, \text{ or} \\
h_i & = c \cdot h_j \text{ for some } 1 \leq j < i \text{ and some } c \in \mathbb{R}.
\end{align*} \]
Show that if the equation \( f(x) = 0 \) can be derived from the equations \( g_1(x) = 0, \ldots, g_n(x) = 0 \), then for any \( v \in \mathbb{R}^\ell, f(v) = 0 \) provided that \( g_1(v) = \cdots = g_n(v) = 0 \).