1. (10 points) Show that $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$ for all integers $n \geq 1$. 
2. (10 points) Let $a_0 = 2$, $a_1 = 5$, and $a_n = 5a_{n-1} - 6a_{n-2}$ for all integers $n \geq 2$. Show that $a_n = 3^n + 2^n$ for all integers $n \geq 0$. 
3. (10 points) Let $n$ be a positive integer and $A_1, \ldots, A_n$ be some sets. Let us define union of these sets as follows:

1. $\cap_{i=1}^1 A_i = A_1$,
2. $\cap_{i=1}^{k+1} A_i = (\cap_{i=1}^k A_i) \cap A_{k+1}$.

Show that $\cap_{i=1}^n \{x \in \mathbb{N} : i \leq x \leq n\} = \{n\}$. 

4. (10 points) Let $U$ be the set of sequences of the following symbols: “+”, “.”, “$x_1$”, . . . , “$x_n$”.

Let $B = \{ x_i : i \in [n] \}$; i.e., $B$ is the set of sequences consisting of only one symbol $x_i$. Let $\mathcal{F} = \{ f_+, f_\cdot \}$, where $f_+(F_1, F_2) = (F_1 + F_2)$ and $f_+(F_1, F_2) = (F_1 \cdot F_2)$ (by $(F_1 \# F_2)$ we denote the sequence obtained by concatenating “("", $F_1$, “#”, $F_2$, and ")”). Let $S$ be the set generated by $\mathcal{F}$ from $B$.

For $s : [n] \to \{0, 1\}$ and $F \in S$, we define the function $\text{val}(F, s)$ using structural recursion as follows.

1. $\text{val}(x_i, s) = s(i),$
2. $\text{val}((F_1 + F_2), s) = \text{val}(F_1, s) + \text{val}(F_2, s),$
3. $\text{val}((F_1 \cdot F_2), s) = \text{val}(F_1, s) \cdot \text{val}(F_2, s).$

Let $F_1, \ldots, F_n \in S$. Let us define the sum of these formulas as follows:

1. $\sum_{i=j}^{j} F_i = F_j,$
2. $\sum_{i=j}^{j+k} F_i = f_+(\sum_{i=j}^{j+k-1} F_i, F_{j+k})$ for $k \geq 1.$

Show that $\text{val}(\sum_{i=1}^{n} x_i, s) = \text{val}(\sum_{i=1}^{n} x_{n-i+1}, s)$ for any $s.$