1. (10 points) Let us formulate the pigeonhole principle using propositional formulas. \( \Omega = \{x_{1,1}, \ldots, x_{n+1,1}, x_{1,2}, \ldots, x_{n+1,n}\} \) (informally \( x_{i,j} \) is true iff the \( i \)th pigeon is in the \( j \)th hole). Consider the following propositional formulas on the variables from \( \Omega \).

- \( L_i \) (\( i \in [n+1] \)) is equal to \( \bigvee_{j=1}^{n} x_{i,j} \). (Informally this formula says that the \( i \)th pigeon is in a hole.)
- \( R_j \) (\( j \in [n] \)) is equal to \( \bigvee_{i_1=1}^{n+1} \bigvee_{i_2=i_1+1}^{n+1} (x_{i_1,j} \land x_{i_2,j}) \). (Informally this formula says that there are two pigeons in the \( j \)th hole.)

Show that there is a natural deduction proof of \( (\bigwedge_{i=1}^{n+1} L_i) \implies (\bigvee_{i=1}^{n} R_i) \).
2. (10 points) Let \( \phi = \bigvee_{i=1}^{m} \lambda_i \) be a clause; we say that the width of the clause is equal to \( m \). Let \( \phi = \bigwedge_{i=1}^{l} \chi_i \) be a formula in CNF; we say that the width of \( \phi \) is equal to the maximal width of \( \chi_i \) for \( i \in [l] \).

Let \( p_n : \{T,F\}^n \to \{T,F\} \) such that \( p_n(x_1, \ldots, x_n) = T \) iff the set \( \{i : x_i = T\} \) has odd number of elements. Show that any CNF representation of \( p_n \) has width at least \( n \).
3. (10 points) Write a natural deduction derivation of \((W \lor Y) \implies (X \lor Z)\) from hypotheses \(W \implies X\) and \(Y \implies Z\).
4. (10 points) We say that a clause $C$ can be obtained from clauses $A$ and $B$ using the *resolution* rule if $C = A' \lor B'$, $A = x \lor A'$, and $B = \neg x \lor B'$, for some variable $x$.

We say that a clause $C$ can be derived from clauses $A_1, \ldots, A_m$ using resolutions if there is a sequence of clauses $D_1, \ldots, D_\ell = C$ such that each $D_i$

- is either obtained from clauses $D_j$ and $D_k$ for $j, k < i$ using the *resolution* rule, or
- is equal to $A_j$ for some $j \in [m]$, or
- is equal to $D_j \lor E$ for some $j < i$ and a clause $E$.

Show that if $A_1, \ldots, A_m$ semantically imply $C$, then $C$ can be derived from clauses $A_1, \ldots, A_m$ using resolutions.