1. (10 points) Show that \(1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}\) for all integers \(n \geq 1\).

**Solution:** We prove the statement using induction by \(n\). The base case for \(n = 1\) is true since \(1^2 = 1\) and \(\frac{1(1+1)(2+1)}{6} = 1\).

Now we need to prove the induction step from \(n\) to \(n + 1\). Assume that \(1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}\). By the hypothesis, \(1^2 + 2^2 + 3^2 + \cdots + n^2 + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2\). Note that

\[
\frac{n(n+1)(2n+1)}{6} + (n+1)^2 = \frac{n(n+1)(2n+1) + 6n^2 + 12n + 6}{6} = \frac{n^3 + 2n^2 + n^2 + 6n^2 + 12n + 6}{6} = \frac{n^3 + 9n^2 + 13n + 6}{6} = \frac{(n+1)(n+2)(2n+3)}{6}.
\]

Hence, the induction step is true and by the induction principle, \(1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}\) for all positive integers \(n\).
2. (10 points) Let \(a_0 = 2\), \(a_1 = 5\), and \(a_n = 5a_{n-1} - 6a_{n-2}\) for all integers \(n \geq 2\). Show that \(a_n = 3^n + 2^n\) for all integers \(n \geq 0\).

**Solution:** We prove the statement using induction by \(n\) for \(n \geq 0\). The base cases for \(n = 0\) and \(n = 1\) are true since \(3^0 + 2^0 = 1 + 1 = 2 = a_0\) and \(3^1 + 2^1 = 3 + 2 = 5 = a_1\).

Let us now prove the induction step from \(n\) and \(n - 1\) to \(n + 1\). Assume that \(a_{n-1} = 3^{n-1} + 2^{n-1}\) and \(a_n = 3^n + 2^n\). Note that \(a_{n+1} = 5a_n - 6a_{n-1}\); hence, by the induction hypothesis, \(a_{n+1} = 5(3^n + 2^n) - 6(3^{n-1} + 2^{n-1}) = (5 \cdot 3 - 6)3^{n-1} + (5 \cdot 2 - 6)2^{n-1} = 9 \cdot 3^{n-1} + 4 \cdot 2^{n-1} = 3^{n+1} + 2^{n+1}\).

As a result, the induction step is true and by the induction principle, \(a_n = 3^n + 2^n\) for all integers \(n \geq 0\). (Note that we proved a stronger statement than it was asked in the problem.)
3. (10 points) Let \( n \) be a positive integer and \( A_1, \ldots, A_n \) be some sets. Let us define union of these sets as follows:

1. \( \bigcap_{i=1}^{1} A_i = A_1 \),
2. \( \bigcap_{i=1}^{k+1} A_i = (\bigcap_{i=1}^{k} A_i) \cap A_{k+1} \).

Show that \( \bigcap_{i=1}^{n} \{x \in \mathbb{N} : i \leq x \leq n\} = \{n\} \).

**Solution:** We prove using induction by \( m \) that \( \bigcap_{i=1}^{m} \{x \in \mathbb{N} : i \leq x \leq n\} = \{m, m+1, \ldots, n\} \).

The base case is for \( m = 1 \) is true since

\[
\bigcap_{i=1}^{1} \{x \in \mathbb{N} : i \leq x \leq n\} = [n] = \{1, 2, \ldots, n\}.
\]

Let us now prove the induction step from \( m \) to \( m+1 \). Assume that \( \bigcap_{i=1}^{m} \{x \in \mathbb{N} : i \leq x \leq n\} = \{m, m+1, \ldots, n\} \). Note that

\[
\bigcap_{i=1}^{m+1} \{x \in \mathbb{N} : i \leq x \leq n\} = \left( \bigcap_{i=1}^{m} \{x \in \mathbb{N} : i \leq x \leq n\} \right) \cap \{x \in \mathbb{N} : m+1 \leq x \leq n\}.
\]

Therefore

\[
\bigcap_{i=1}^{m+1} \{x \in \mathbb{N} : i \leq x \leq n\} = \{m, m+1, \ldots, n\} \cap \{m+1, \ldots, n\} = \{m+1, \ldots, n\}.
\]

Hence, by the induction principle, the statement is true for all \( m \). As a result, we proved for \( m = n \) that \( \bigcap_{i=1}^{n} \{x \in \mathbb{N} : i \leq x \leq n\} = \{n\} \).
4. (10 points) Let $U$ be the set of sequences of the following symbols: “+”, “-”, “x_1”, \ldots, “x_n”. Let $B = \{x_i : i \in [n]\}$; i.e., $B$ is the set of sequences consisting of only one symbol $x_i$. Let $F = \{f_+, f\}$, where $f_+(F_1, F_2) = (F_1 + F_2)$ and $f(F_1, F_2) = (F_1, F_2)$ (by $F_1 \# F_2$ we denote the sequence obtained by concatenating “(#”, “x_1”, “…,” “x_n”). Let $S$ be the set generated by $F$ from $B$.

For $s : [n] \to \{0, 1\}$ and $F \in S$, we define the function val($F, s$) using structural recursion as follows.

1. val($x_i, s$) = $s(i)$,
2. val($(F_1 + F_2), s$) = val($F_1, s$) + val($F_2, s$),
3. val($(F_1 \cdot F_2), s$) = val($F_1, s$) \cdot val($F_2, s$).

Let $F_1, \ldots, F_n \in S$. Let us define the sum of these formulas as follows:

1. $\sum_{i=j}^{j} F_i = F_j$,
2. $\sum_{i=j}^{j+k} F_i = f_+((\sum_{i=j}^{j+k-1} F_i, F_{j+k}))$ for $k \geq 1$.

Show that $\text{val}(\sum_{i=1}^{n} x_i, s) = \text{val}(\sum_{i=1}^{n} x_{n-i+1}, s)$ for any $s$.

**Solution:** Before we start working with the arithmetic formulas, let us prove several statements for real number. Let $a_1, \ldots, a_n$ be some real numbers. We show that $\sum_{i=m}^{m+n} a_i = a_m + \sum_{i=m+1}^{m+n} a_i$ for $n \geq 1$ using induction by $n$. The base case is true for $n = 1$ since $\sum_{i=m}^{m+1} a_i = a_m + a_{m+1} = a_m + \sum_{i=m+1}^{m} a_i$.

Let us now prove the induction step from $n$ to $n+1$. Assume that $\sum_{i=m}^{m+n} a_i = a_m + \sum_{i=m+1}^{m+n} a_i$.

Note that by the induction hypothesis,

$$\sum_{i=m}^{m+n+1} a_i = \sum_{i=m}^{m+n} a_i + a_{m+n+1} = a_m + \sum_{i=m+1}^{m+n} a_i + a_{m+n+1} = a_m + \sum_{i=m+1}^{m+n+1} a_i.$$  

Using this statement we may show that $\sum_{i=1}^{m} a_i = \sum_{i=n-m+1}^{n} a_{n-i+1}$ for $m \geq 1$ using induction by $m$. The base case is true since $\sum_{i=1}^{m} a_i = a_1 = \sum_{i=n}^{n} a_{n-i+1}$. To prove the induction step from $m$ to $m+1$; assume $\sum_{i=1}^{m} a_i = \sum_{i=n-m+1}^{n} a_{n-i+1}$. Note that the hypothesis implies that

$$\sum_{i=1}^{m+1} a_i = \sum_{i=1}^{m} a_i + a_{m+1} = \sum_{i=n-m+1}^{n} a_{n-i+1} + a_{m+1} = \sum_{i=n-m}^{n} a_{n-i+1}.$$  

Therefore by the induction hypothesis, $\sum_{i=1}^{m+1} a_i = \sum_{i=n-m+1}^{n} a_{n-i+1}$ for $m \geq 1$. If we consider $m = n$, we get $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} a_{n-i+1}$.

Let us now explain how to get this statement for arithmetic formulas. Let $F_1, \ldots, F_m$ be some arithmetic formulas. Then we may show that $\text{val}(\sum_{i=1}^{m} F_i, s) = \sum_{i=1}^{m} \text{val}(F_i, s)$ for all $s$. Fix some $s$; we prove this statement also using induction. The base case for $m = 1$ is true since $\sum_{i=1}^{1} F_i = F_1$ and $\sum_{i=1}^{1} \text{val}(F_i, s) = \text{val}(F_1, s)$. To prove the induction step from $m$ to $m+1$; assume $\text{val}(\sum_{i=1}^{m} F_i, s) = \sum_{i=1}^{m} \text{val}(F_i, s)$. Note that $\sum_{i=1}^{m+1} F_i = f_+((\sum_{i=1}^{m} F_i, F_{m+1}))$, and $\text{val}(f_+(\sum_{i=1}^{m} F_i, F_{m+1}), s) = \text{val}(\sum_{i=1}^{m} F_i, s) + \text{val}(F_{m+1}, s)$. Hence,

$$\text{val}(\sum_{i=1}^{m+1} F_i, s) = \text{val}(\sum_{i=1}^{m} F_i, s) + \text{val}(F_{m+1}, s) = \sum_{i=1}^{m} \text{val}(F_i, s) + \text{val}(F_{m+1}, s) = \sum_{i=1}^{m+1} \text{val}(F_i, s).$$  

Using all these statement, we may notice that

$$\text{val}(\sum_{i=1}^{n} x_i, s) = \sum_{i=1}^{n} \text{val}(x_i, s) = \sum_{i=1}^{n} \text{val}(x_{n-i+1}, s) = \text{val}(\sum_{i=1}^{n} x_{n-i+1}, s).$$