1. (10 points) Show that \(1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}\) for all integers \(n \geq 1\).

**Solution:** We prove the statement using induction by \(n\). The base case for \(n = 1\) is true since \(1^2 = 1\) and \(\frac{1(1+1)(2+1)}{6} = 1\).

Now we need to prove the induction step from \(n\) to \(n+1\). Assume that \(1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}\). By the hypothesis, \(1^2 + 2^2 + 3^2 + \cdots + n^2 + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2\). Note that

\[
\frac{n(n+1)(2n+1)}{6} + (n+1)^2 = \frac{n(n+1)(2n+1) + 6n^2 + 12n + 6}{6} = \frac{n^3 + 2n^2 + n + 6n^2 + 12n + 6}{6} = \frac{n^3 + 9n^2 + 13n + 6}{6} = \frac{(n+1)(n+2)(2n+3)}{6}.
\]

Hence, the induction step is true and by the induction principle, \(1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}\) for all positive integers \(n\).
2. (10 points) Let \( a_0 = 2 \), \( a_1 = 5 \), and \( a_n = 5a_{n-1} - 6a_{n-2} \) for all integers \( n \geq 2 \). Show that \( a_n = 3^n + 2^n \) for all integers \( n \geq 0 \).

**Solution:** We prove the statement using induction by \( n \) for \( n \geq 0 \). The base cases for \( n = 0 \) and \( n = 1 \) are true since \( 3^0 + 2^0 = 1 + 1 = 2 = a_0 \) and \( 3^1 + 2^1 = 3 + 2 = 5 = a_1 \).

Let us now prove the induction step from \( n \) and \( n - 1 \) to \( n + 1 \). Assume that \( a_{n-1} = 3^{n-1} + 2^{n-1} \) and \( a_n = 3^n + 2^n \). Note that \( a_{n+1} = 5a_n - 6a_{n-1} \); hence, by the induction hypothesis, \( a_{n+1} = 5(3^n + 2^n) - 6(3^{n-1} + 2^{n-1}) = (5 \cdot 3^n - 6 \cdot 3^{n-1}) + (5 \cdot 2^n - 6 \cdot 2^{n-1}) = 9 \cdot 3^{n-1} + 4 \cdot 2^{n-1} = 3^{n+1} + 2^{n+1} \).

As a result, the induction step is true and by the induction principle, \( a_n = 3^n + 2^n \) for all integers \( n \geq 0 \). (Note that we proved a stronger statement than it was asked in the problem.)
3. (10 points) Let $n$ be a positive integer and $A_1, \ldots, A_n$ be some sets. Let us define union of these sets as follows:

1. $\cap_{i=1}^1 A_i = A_1$,
2. $\cap_{i=1}^{k+1} A_i = (\cap_{i=1}^k A_i) \cap A_{k+1}$.

Show that $\cap_{i=1}^n \{x \in \mathbb{N} : i \leq x \leq n\} = \{n\}$.

**Solution:** We prove using induction by $m$ that $\cap_{i=1}^m \{x \in \mathbb{N} : i \leq x \leq n\} = \{m, m+1, \ldots, n\}$.

The base case is for $m = 1$ is true since

$$\cap_{i=1}^1 \{x \in \mathbb{N} : i \leq x \leq n\} = [n] = \{1, 2, \ldots, n\}.$$

Let us now prove the induction step from $m$ to $m+1$. Assume that $\cap_{i=1}^m \{x \in \mathbb{N} : i \leq x \leq n\} = \{m, m+1, \ldots, n\}$. Note that

$$\cap_{i=1}^{m+1} \{x \in \mathbb{N} : i \leq x \leq n\} = (\cap_{i=1}^m \{x \in \mathbb{N} : i \leq x \leq n\}) \cap \{x \in \mathbb{N} : m+1 \leq x \leq n\}.$$

Therefore

$$\cap_{i=1}^{m+1} \{x \in \mathbb{N} : i \leq x \leq n\} = \{m, m+1, \ldots, n\} \cap \{m+1, \ldots, n\} = \{m+1, \ldots, n\}.$$

Hence, by the induction principle, the statement is true for all $m$. As a result, we proved for $m = n$ that $\cap_{i=1}^n \{x \in \mathbb{N} : i \leq x \leq n\} = \{n\}$. 

4. (10 points) Let $U$ be the set of sequences of the following symbols: “$+$”, “$\cdot$”, “$x_1$”, . . ., “$x_n$”. Let $B = \{x_i : i \in [n]\}$; i.e., $B$ is the set of sequences consisting of only one symbol $x_i$. Let $F = \{f_+, f_\cdot\}$, where $f_+(F_1, F_2) = (F_1 + F_2)$ and $f_+(F_1, F_2) = (F_1 \cdot F_2)$ (by $(F_1 \# F_2)$ we denote the sequence obtained by concatenating “($+$”, “$\cdot$”, “$#$”, “$F_2$”, and “$+$”)”). Let $S$ be the set generated by $F$ from $B$.

For $s : [n] \to \{0, 1\}$ and $F \in S$, we define the function $\text{val}(F, s)$ using structural recursion as follows.

1. $\text{val}(x_i, s) = s(i)$,
2. $\text{val}((F_1 + F_2), s) = \text{val}(F_1, s) + \text{val}(F_2, s)$,
3. $\text{val}((F_1 \cdot F_2), s) = \text{val}(F_1, s) \cdot \text{val}(F_2, s)$.

Let $F_1, \ldots, F_n \in S$. Let us define the sum of these formulas as follows:

1. $\sum_{i=j}^{\infty} F_i = F_j$,
2. $\sum_{i=j}^{\infty+k} F_i = f_+(\sum_{i=j}^{\infty+k-1} F_i, F_{j+k})$ for $k \geq 1$.

Show that $\text{val}(\sum_{i=1}^{n} x_i, s) = \text{val}(\sum_{i=1}^{n} x_{n-i+1}, s)$ for any $s$.

\[ \sum_{i=m}^{m+n+1} a_i = \sum_{i=m}^{m+n} a_i + a_{m+n+1} = a_m + \sum_{i=m+1}^{m+n} a_i + a_{m+n+1} = a_m + \sum_{i=m+1}^{m+n+1} a_i. \]

Using this statement we may show that $\sum_{i=1}^{m} a_i = \sum_{i=n-m}^{n} a_{n-i+1}$ for $m \geq 1$ using induction by $m$. The base case is true for $n = 1$ since $\sum_{i=1}^{1} a_i = a_m + a_{m+1} = a_m + \sum_{i=m+1}^{m+n} a_i$.

Let us now prove the induction step from $n$ to $n + 1$. Assume that $\sum_{i=m}^{m+n} a_i = a_m + \sum_{i=m+1}^{m+n} a_i$.

Note that by the induction hypothesis,

\[ \sum_{i=m}^{m+n+1} a_i = \sum_{i=m}^{m+n} a_i + a_{m+n+1} = a_m + \sum_{i=m+1}^{m+n} a_i + a_{m+n+1} = a_m + \sum_{i=m+1}^{m+n+1} a_i. \]

Using this statement we may show that $\sum_{i=1}^{k} a_i = \sum_{i=n-k}^{n} a_{n-i+1}$ for $k \geq 1$ using induction by $m$. The base case is true since $\sum_{i=1}^{1} a_i = a_1 = \sum_{i=n}^{n} a_{n-i+1}$. To prove the induction step from $m$ to $m + 1$; assume $\sum_{i=1}^{k} a_i = \sum_{i=n-k}^{n} a_{n-i+1}$. Note that the hypothesis implies that

\[ \sum_{i=1}^{m+1} a_i = \sum_{i=1}^{m} a_i + a_{m+1} = \sum_{i=n-m}^{n} a_{n-i+1} + a_{m+1} = \sum_{i=n-m}^{n} a_{n-i+1}. \]

Therefore by the induction hypothesis, $\sum_{i=1}^{m+1} a_i = \sum_{i=n-m}^{n} a_{n-i+1}$ for $m \geq 1$. If we consider $m = n$, we get $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} a_{n-i+1}$.

Let us now explain how to get this statement for arithmetic formulas. Let $F_1, \ldots, F_m$ be some arithmetic formulas. Then we may show that $\text{val}(\sum_{i=1}^{m} F_i, s) = \sum_{i=1}^{m} \text{val}(F_i, s)$ for all $s$.

Fix some $s$; we prove this statement also using induction. The base case for $m = 1$ is true since $\sum_{i=1}^{1} F_i = F_1$ and $\sum_{i=1}^{1} \text{val}(F_i, s) = \text{val}(F_1, s)$. To prove the induction step from $m$ to $m + 1$; assume $\text{val}(\sum_{i=1}^{m} F_i, s) = \sum_{i=1}^{m} \text{val}(F_i, s)$. Note that $\sum_{i=1}^{m+1} F_i = f_+(\sum_{i=1}^{m} F_i, F_{m+1})$, and $\text{val}(f_+(\sum_{i=1}^{m} F_i, F_{m+1}), s) = \text{val}(\sum_{i=1}^{m} F_i, s) + \text{val}(F_{m+1}, s)$. Hence,

\[ \text{val}(\sum_{i=1}^{m+1} F_i, s) = \text{val}(\sum_{i=1}^{m} F_i, s) + \text{val}(F_{m+1}, s) = \sum_{i=1}^{m} \text{val}(F_i, s) + \text{val}(F_{m+1}, s) = \sum_{i=1}^{m+1} \text{val}(F_i, s). \]

Using all these statements, we may notice that

\[ \text{val}(\sum_{i=1}^{n} x_i, s) = \sum_{i=1}^{n} \text{val}(x_i, s) = \sum_{i=1}^{n} \text{val}(x_{n-i+1}, s) = \sum_{i=1}^{n} \text{val}(x_{n-i+1}, s). \]