1. (10 points) Let $S \subseteq \mathbb{N}$ be a nonempty set. Show that $S$ is decidable iff there is a function $f : \mathbb{N} \to \mathbb{N}$ such that $f$ is computable, $f$ is nondecreasing, and $\text{Im } f = S$.

**Solution:** Let us assume that $S$ is decidable and $A$ decides $S$. Let $x_{\text{min}}$ be the minimal element of $S$. Consider the following algorithm.

1: function $F(n)$
2: if $n < x_{\text{min}}$ then
3: return $x_{\text{min}}$
4: end if
5: Let $x \leftarrow n$
6: while $\neg A(x)$ and $x > x_{\text{min}}$ do
7: $x \leftarrow x - 1$
8: end while
9: return $x$
10: end function

It is clear that this algorithm computes the total function $f : \mathbb{N} \to \mathbb{N}$ such that 

$$f(x) = \begin{cases} x_{\text{min}} & \text{if } x < x_{\text{min}} \\ \max\{y \in S : y \leq x\} & \text{otherwise} \end{cases}$$

Therefore $f$ is nondecreasing. We need to prove now that $\text{Im } f = S$. To prove this first we note that it is easy to see that $\text{Im } f \subseteq S$. In addition, if $x \in S$, then $f(x) = x$, which implies that $\text{Im } f = S$.

Let us now prove the statement in the opposite direction. Assume that there is a total nondecreasing function $f : \mathbb{N} \to \mathbb{N}$ such that $\text{Im } f = S$.

- If $S$ is finite, then we prove in class that it is decidable.
- Let $S$ be an infinite set, and let $F$ be an algorithm computing $f$. Consider the following algorithm

1: function $A(x)$
2: while $F(n) < x$ do
3: $n \leftarrow n + 1$
4: end while
5: if $F(n) = x$ then
6: return 1
7: else
8: return 0
9: end if
10: end function

In lines 2-4 this algorithm finds the minimal $n$ such that $f(n) \geq x$ (it always exists since $S$ is infinite). If $x \in S$, then such a minimal $n$ is equal $x$, and the algorithm returns $1$. Otherwise $f(n) \neq x$ and the algorithm returns $0$. 
2. (10 points) Let $A, B \subseteq \mathbb{N}$ be enumerable sets. Show that $A \times B$ is enumerable.

\textbf{Solution:} Let $A$ and $B$ be semideciding algorithms for $A$ and $B$, respectively. Consider the following algorithm.

1: \textbf{function} $C(x, y)$
2: \hspace{1em} $A(x)$
3: \hspace{1em} $B(x)$
4: \hspace{1em} \textbf{return} 1
5: \textbf{end function}

Note that if $x \in A$ and $y \in B$, then the algorithm return 1. Otherwise the algorithm never terminates. Therefore, if $(x, y) \in A \times B$, then the algorithm return 1. Otherwise the algorithm never terminates.