1. Let $\ell_1, \ldots, \ell_k$ be some nonnegative numbers such that $\ell_1 + \cdots + \ell_k = \ell$. Find the number of weak compositions (in terms of $\ell$, $k$, and $n$) $(a_1, \ldots, a_k)$ of $n$ into $k$ such that $a_i \geq \ell_i$.

**Solution:** Note that the problem is equivalent to the problem of placing $n$ identical balls into $k$ different boxes so that $i$th box has at least $\ell_i$ balls inside.

Let us put in advance $\ell_i$ balls into the $i$th box for all $i \in [n]$. After this we have $n - \ell$ balls left to place into $k$ boxes and there are $\binom{n-\ell+k-1}{k-1}$ ways to do this.
2. Let $n$ be a natural number.

(a) Find an explicit formula for $S(n, n-2)$.

**Solution:** Note that there are two types of partition of $[n]$ into $n-2$ parts:

1. all the parts have size 1 except two of size 2,
2. all the parts have size 1 except one that has size 3.

It is easy to see that there are $\binom{n}{3}$ partitions of the second kind, and there are $\frac{1}{2} \binom{n}{2} (n-2)$ partitions of the first kind. Therefore $S(n, n-2) = \frac{1}{2} \binom{n}{2} (n-2) + \binom{n}{3}$.

(b) Find an explicit formula for $S(n, 3)$.

**Solution:** In this problem it is easier to find the number of surjections from $[n]$ to $[3]$. There are $3^n$ functions from $[n]$ to $[3]$, there are $3 \cdot 2^n$ functions from $[n]$ to $[3]$ such that the image has 2 elements, and there are 3 functions such that their image has 1 element. Therefore, by the inclusion-exclusion principle, there are $3^n - 3 \cdot 2^n + 3$ surjections from $[n]$ to $[3]$. As a result, $S(n, 3) = \frac{1}{3}(3^n - 3 \cdot 2^n + 3)$. 
3. How many numbers must be selected from the set \([6]\) to guarantee that at least one pair of these numbers add up to 7?

**Solution:** Let us split the numbers from \([6]\) into 3 pairs:

1. 1 and 6,
2. 2 and 5,
3. 3 and 4.

Note that the sum within the pair is 7. It is also easy to see that, by the pigeonhole principle, if we pick 4 numbers out of \([6]\), two of them are in the same pair. Therefore whenever we choose 4 numbers two of them sum up to 7.

It is also easy to see that we may select 3 numbers so that the sum of any two of them is not equal to 7. For example, we may select 1, 2, and 3.
4. Show that \( \int_{0}^{\infty} x^n e^{-x} \, dx = n! \) for all \( n \geq 0 \).

**Solution:** We prove the statement using induction by \( n \). The base case is for \( n = 0 \). It is easy to see that \( \int_{0}^{\infty} e^{-x} \, dx = (e^{-x})|_{0}^{\infty} = 1 \).

Let us prove the induction step from \( k \) to \( k + 1 \). By the induction hypothesis, \( \int_{0}^{\infty} x^k e^{-x} \, dx = k! \). Note that

\[
\int_{0}^{\infty} x^{k+1} e^{-x} \, dx = -x^{k+1} e^{-x}|_{0}^{\infty} + \int_{0}^{\infty} (k+1)x^k e^{-x} \, dx = \int_{0}^{\infty} (k+1)x^k e^{-x} \, dx = (k+1)!.
\]