1. Alice and Bob play the following game.
   - Initially, there are 20 numbers: 10 numbers 1 and 10 numbers 2.
   - On each step one of the players select two numbers; and if they were the same, replace them by 2; otherwise, replace them by 1.
   - Alice make the first move and they do moves one after another.

Alice wins if the last number is 1 and Bob wins if the last number is 2. Who is the winner?

Note that this game is not a combinatorial game.

Solution: Note that after each step the reminder modulo 2 of the sum of numbers is preserved e.g if we replace 1 and 1 by 2 the sum has the same reminder modulo 2. Initially, the sum is even; hence, at the end we will have 2 and Bob wins.
2. In the subtraction game where players may subtract 1, 2 or 5 chips on their turn, identify the N and P positions.

Solution: We are going to prove that position $n$ is a P-position iff $n \equiv 0 \pmod{3}$. We prove this statement using the strong induction by $n$.

The base case is for $n \in \{0, 1, 2\}$. Note that since $n = 0$ is a terminal position it is a P-position and from 1 and 2 we can go to 0 i.e. they are N-positions.

Let us prove now the induction step: by induction hypothesis we know that for every $k < n$: $k$ is a P-position iff $k \equiv 0 \pmod{3}$.

- If $n \equiv 0 \pmod{3}$, then we can go to $n - 1$, $n - 2$ and $n - 5$ and all of them (by the induction hypothesis) are N-positions. Hence, $n$ is a P-position.

- If $n \not\equiv 0 \pmod{3}$, then there is $r \in \{1, 2\}$ such that $n - r \equiv 0 \pmod{3}$ and we can go to $n - r$ which is a P-position. Hence, $n$ is an N-position.
3. Is the Nim position $(1, 3, 5)$ an N-position (explain your answer)?

**Solution:** First of all note that $(0, 2, 2)$, $(1, 1, 0)$, and $(1, 0, 1)$ are P-positions by the theorem about two-pile Nim. Hence, $(1, 2, 2)$, $(1, 1, 2)$, $(1, 0, 2)$, and $(1, 3, 1)$ are N-positions. As a result, $(1, 3, 2)$ is a P-position since $(0, 3, 2)$ and $(1, 3, 0)$ are N-positions. Finally, it implies that $(1, 3, 5)$ is an N-position.
4. Consider the Misère subtraction game where players may subtract 1, 5 or 6 chips on their turn, identify the N and P positions.

**Solution:** We are going to prove that position $n$ is a P-position iff the remainder of $n$ modulo 11 belongs to \{1, 3, 5\}. We prove this statement using the strong induction by $n$.

The base case is for $n \in \{0, 1, \ldots, 10\}$. Note that since $n = 0$ is a terminal position it is an N-position. From 1 we can go only to 0 i.e. it is a P-position. Since there is a move from 2 to 1 it is an N-state etc. Hence, types of states can be represented by the following table.

<table>
<thead>
<tr>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>N</td>
<td>P</td>
<td>N</td>
<td>P</td>
<td>N</td>
<td>P</td>
<td>N</td>
<td>N</td>
<td>N</td>
<td>N</td>
<td>N</td>
</tr>
</tbody>
</table>

Let us prove now the induction step: by induction hypothesis we know that for every $k < n$: $k$ is a P-position iff the remainder of $k$ modulo 11 belongs to \{1, 3, 5\}.

- If $n \equiv 0 \pmod{11}$, then we can go to $n - 6$ and the reminder modulo 11 is 5; hence, $n$ is an N-position.
- If $n \equiv 1 \pmod{11}$, then we can only go to $n - 1, n - 5$, and $n - 6$ and the reminders modulo 11 are 1, 7, and 6; hence, $n$ is a P-position.
- If $n \equiv 2 \pmod{11}$, then we can go to $n - 1$ and the reminder modulo 11 is 1; hence, $n$ is an N-position.
- If $n \equiv 3 \pmod{11}$, then we can only go to $n - 1, n - 5$, and $n - 6$ and the reminders modulo 11 are 2, 9, and 8; hence, $n$ is a P-position.
- If $n \equiv 4 \pmod{11}$, then we can go to $n - 1$ and the reminder modulo 11 is 3; hence, $n$ is an N-position.
- If $n \equiv 5 \pmod{11}$, then we can only go to $n - 1, n - 5$, and $n - 6$ and the reminders modulo 11 are 4, 0, and 10; hence, $n$ is a P-position.
- If $n \equiv 6 \pmod{11}$, then we can go to $n - 1$ and the reminder modulo 11 is 5; hence, $n$ is an N-position.
- If $n \equiv 7 \pmod{11}$, then we can go to $n - 6$ and the reminder modulo 11 is 1; hence, $n$ is an N-position.
- If $n \equiv 8 \pmod{11}$, then we can go to $n - 5$ and the reminder modulo 11 is 3; hence, $n$ is an N-position.
- If $n \equiv 9 \pmod{11}$, then we can go to $n - 6$ and the reminder modulo 11 is 3; hence, $n$ is an N-position.
- If $n \equiv 10 \pmod{11}$, then we can go to $n - 5$ and the reminder modulo 11 is 5; hence, $n$ is an N-position.