1. Alice and Bob play the following game.

- Initially, there are 20 numbers: 10 numbers 1 and 10 numbers 2.
- On each step one of the players select two numbers; and if they were the same, replace them by 2; otherwise, replace them by 1.
- Alice make the first move and they do moves one after another.

Alice wins if the last number is 1 and Bob wins if the last number is 2. Who is the winner?

**Note that this game is not a combinatorial game.**

**Solution:** Note that after each step the reminder modulo 2 of the sum of numbers is preserved e.g if we replace 1 and 1 by 2 the sum has the same reminder modulo 2. Initially, the sum is even; hence, at the end we will have 2 and Bob wins.
2. In the subtraction game where players may subtract 1, 2 or 5 chips on their turn, identify the N and P positions.

**Solution:** We are going to prove that position \( n \) is a P-position iff \( n \equiv 0 \pmod{3} \). We prove this statement using the strong induction by \( n \).

The base case is for \( n \in \{0, 1, 2\} \). Note that since \( n = 0 \) is a terminal position it is a P-position and from 1 and 2 we can go to 0 i.e. they are N-positions.

Let us prove now the induction step: by induction hypothesis we know that for every \( k < n \): \( k \) is a P-position iff \( k \equiv 0 \pmod{3} \).

- If \( n \equiv 0 \pmod{3} \), then we can go to \( n - 1, n - 2 \) and \( n - 5 \) and all of them (by the induction hypothesis) are N-positions. Hence, \( n \) is a P-position.

- If \( n \not\equiv 0 \pmod{3} \), then there is \( r \in \{1, 2\} \) such that \( n - r \equiv 0 \pmod{3} \) and we can go to \( n - r \) which is a P-position. Hence, \( n \) is an N-position.
3. Is the Nim position $(1, 3, 5)$ an N-position (explain your answer)?

**Solution:** First of all note that $(0, 2, 2)$, $(1, 1, 0)$, and $(1, 0, 1)$ are P-positions by the theorem about two-pile Nim. Hence, $(1, 2, 2)$, $(1, 1, 2)$, $(1, 0, 2)$, and $(1, 3, 1)$ are N-positions. As a result, $(1, 3, 2)$ is a P-position since $(0, 3, 2)$ and $(1, 3, 0)$ are N-positions. Finally, it implies that $(1, 3, 5)$ is an N-position.
4. Consider the Misère subtraction game where players may subtract 1, 5 or 6 chips on their turn, identify the N and P positions.

**Solution:** We are going to prove that position \( n \) is a P-position iff the remainder of \( n \) modulo 11 belongs to \( \{1, 3, 5\} \). We prove this statement using the strong induction by \( n \).

The base case is for \( n \in \{0, 1, \ldots, 10\} \). Note that since \( n = 0 \) is a terminal position it is an N-position. From 1 we can go only to 0 i.e. it is a P-position. Since there is a move from 2 to 1 it is an N-state etc. Hence, types of states can be represented by the following table.

<table>
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<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
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</thead>
<tbody>
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<td>P</td>
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<td>P</td>
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<td>P</td>
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<td>N</td>
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<td>N</td>
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</tr>
</tbody>
</table>

Let us prove now the induction step: by induction hypothesis we know that for every \( k < n \): \( k \) is a P-position iff the remainder of \( k \) modulo 11 belongs to \( \{1, 3, 5\} \).

- If \( n \equiv 0 \pmod{11} \), then we can go to \( n - 6 \) and the reminder modulo 11 is 5; hence, \( n \) is an N-position.
- If \( n \equiv 1 \pmod{11} \), then we can only go to \( n - 1, n - 5, \) and \( n - 6 \) and the reminders modulo 11 are 1, 7, and 6; hence, \( n \) is a P-position.
- If \( n \equiv 2 \pmod{11} \), then we can go to \( n - 1 \) and the reminder modulo 11 is 1; hence, \( n \) is an N-position.
- If \( n \equiv 3 \pmod{11} \), then we can only go to \( n - 1, n - 5, \) and \( n - 6 \) and the reminders modulo 11 are 2, 9, and 8; hence, \( n \) is a P-position.
- If \( n \equiv 4 \pmod{11} \), then we can go to \( n - 1 \) and the reminder modulo 11 is 3; hence, \( n \) is an N-position.
- If \( n \equiv 5 \pmod{11} \), then we can only go to \( n - 1, n - 5, \) and \( n - 6 \) and the reminders modulo 11 are 4, 0, and 10; hence, \( n \) is a P-position.
- If \( n \equiv 6 \pmod{11} \), then we can go to \( n - 1 \) and the reminder modulo 11 is 5; hence, \( n \) is an N-position.
- If \( n \equiv 7 \pmod{11} \), then we can go to \( n - 6 \) and the reminder modulo 11 is 1; hence, \( n \) is an N-position.
- If \( n \equiv 8 \pmod{11} \), then we can go to \( n - 5 \) and the reminder modulo 11 is 3; hence, \( n \) is an N-position.
- If \( n \equiv 9 \pmod{11} \), then we can go to \( n - 6 \) and the reminder modulo 11 is 3; hence, \( n \) is an N-position.
- If \( n \equiv 10 \pmod{11} \), then we can go to \( n - 5 \) and the reminder modulo 11 is 5; hence, \( n \) is an N-position.