1. Compute Grundy function for states of the subtraction game where players may subtract 1, 2 or 5 chips on their turn.

**Solution:** Let $g(x)$ be the Grundy function for the position $x$ in this game.
We claim that $g(x) = x \mod 3$ for all nonnegative integers $x$.

We prove this statement as using induction, as usual. Base cases:

- $x = 0$ : $g(0) = 0$, since it is a terminal position;
- $x = 1$ : $g(1) = \text{mex } g(0) = \text{mex } 0 = 1$;
- $x = 2$ : $g(2) = \text{mex}\{g(0), g(1)\} = \text{mex}\{0, 1\} = 2$.

The inductive hypothesis is that $g(x) = x \mod 3$ for all non-negative integers $x \leq n$, where $n$ is a non-negative integer. We need to show that $g(n + 1) = (n + 1) \mod 3$. Note that

$$g(n + 1) = \text{mex}\{g(n), g(n - 1), g(n - 4)\} = \text{mex}\{n \mod 3, (n - 1) \mod 3, (n - 4) \mod 3\} = \text{mex}\{n \mod 3, (n + 2) \mod 3\}.$$  

Since $n \mod 3$ and $(n + 2) \mod 3$ take two different values in $\{0, 1, 2\}$, then $(n + 1) \mod 3 = \{0, 1, 2\} \{n \mod 3, (n + 2) \mod 3\}$

Hence, the Grundy function $g(x) = x \mod 3$ for any position $x$ in the subtraction game where players may subtract 1, 2 or 5 chips on their turn.
2. Compute Grundy function for the Nim position $(1, 3, 5)$.

**Solution:** It is easy to solve this problem using brute force, we try to decrease number of cases. We note that reordering of the piles does not change the value of the Grundy function and if the only non-zero pile has $a$ pebbles, then the value is equal to $a$. Additionally, if the position is a P-poistion, then the value is 0.

- $g(0, 1, 2) = \text{mex}\{g(0, 0, 2), g(0, 1, 1), g(0, 0, 1)\} = 3$;
- $g(0, 1, 3) = \text{mex}\{g(0, 0, 3), g(0, 1, 2), g(0, 1, 1), g(0, 0, 1)\} = \text{mex}\{3, g(0, 1, 2), 0, 1\} = 2$;
- $g(0, 1, 4) = \text{mex}\{g(0, 0, 1), g(0, 1, 1), g(0, 1, 2), g(0, 1, 3), g(0, 0, 4)\} = 5$;
- $g(0, 1, 5) = \text{mex}\{g(0, 0, 1), g(0, 1, 1), g(0, 1, 2), g(0, 1, 3), g(0, 1, 4), g(0, 0, 5)\} = 4$;
- $g(0, 2, 3) = \text{mex}\{g(0, 2, 2), g(0, 1, 2), g(0, 0, 2), g(0, 1, 3), g(0, 0, 3)\} = 1$;
- $g(0, 2, 4) = \text{mex}\{g(0, 2, 3), g(0, 2, 2), g(0, 1, 2), g(0, 0, 2), g(0, 1, 4), g(0, 0, 4)\} = 6$;
- $g(0, 2, 5) = \text{mex}\{g(0, 2, 4), g(0, 2, 3), g(0, 2, 2), g(0, 1, 2), g(0, 0, 2), g(0, 1, 5), g(0, 0, 5)\} = 7$;
- $g(0, 3, 4) = \text{mex}\{g(0, 2, 4), g(0, 1, 4), g(0, 0, 4), g(0, 3, 3), g(0, 2, 3), g(0, 1, 3), g(0, 0, 3)\} = 7$;
- $g(0, 3, 5) = \text{mex}\{g(0, 2, 5), g(0, 1, 5), g(0, 0, 5), g(0, 3, 4), g(0, 3, 3), g(0, 2, 3)g(0, 1, 3), g(0, 0, 3)\} = 6$;
- $g(1, 1, 1) = \text{mex}\{g(1, 1, 0), g(1, 0, 1), g(0, 1, 1)\} = 1$;
- $g(1, 1, 2) = \text{mex}\{g(0, 1, 1), g(1, 1, 1), g(0, 1, 2)\} = 2$;
- $g(1, 1, 3) = \text{mex}\{g(0, 1, 3), g(1, 1, 2), g(1, 1, 1), g(0, 1, 1)\} = 3$;
- $g(1, 1, 4) = \text{mex}\{g(0, 1, 4), g(1, 1, 3), g(1, 1, 2), g(1, 1, 1), g(0, 1, 1)\} = 4$;
- $g(1, 1, 5) = \text{mex}\{g(0, 1, 5), g(1, 1, 4), g(1, 1, 3), g(1, 1, 2), g(1, 1, 1), g(0, 1, 1)\} = 5$;
- $g(1, 2, 4) = \text{mex}\{g(0, 2, 4), g(1, 1, 4), g(0, 1, 4), g(0, 2, 3), g(0, 2, 2), g(0, 1, 2), g(0, 0, 2)\} = 7$;
- $g(1, 3, 3) = \text{mex}\{g(0, 3, 3), g(1, 3, 2), g(1, 3, 1), g(1, 3, 0)\} = 1$;
- $g(1, 2, 5) = \text{mex}\{g(0, 2, 5), g(1, 1, 5), g(0, 1, 5), g(1, 2, 4), g(1, 2, 3), g(1, 2, 2), g(1, 1, 2), g(0, 1, 2)\} = 6$;
- $g(1, 3, 4) = \text{mex}\{g(0, 3, 4), g(1, 2, 4), g(1, 1, 4), g(0, 1, 4), g(1, 3, 3), g(1, 3, 2), g(1, 1, 3), g(0, 1, 3)\} = 6$;
- $g(1, 3, 5) = \text{mex}\{g(0, 3, 5), g(1, 2, 5), g(1, 1, 5), g(0, 1, 5), g(1, 3, 4), g(1, 3, 3), g(1, 3, 2), g(1, 1, 3), g(0, 1, 3)\} = 7$. 

3. Consider the following game: Alice and Bob are writing from left to right digits of a 11-digit number one by one. Alice wins if the number divides by 7 and Bob wins otherwise (Alice writes the first the digit). Determine who is the winner.

Solution: Since Alice writes the first digit, Alice will write the last digit too. Let the 11-digit number be $a_1a_2a_3a_4a_5a_6a_7a_8a_9a_{10}a_{11}$, where $a_i$ denotes the $i$th digit (from left to right). Suppose $a_1$ to $a_{10}$ are already determined, let $a_{11} = 7 - \frac{a_1a_2a_3a_4a_5a_6a_7a_8a_9a_{10}}{10} \mod 7$. Then $a_1a_2a_3a_4a_5a_6a_7a_8a_9a_{10}a_{11} \mod 7 = (a_1a_2a_3a_4a_5a_6a_7a_8a_9a_{10} \cdot 10 + a_{11}) \mod 7 = 0$. Thus Alice is the winner because she can always manage to make the number divisible by 7.