1. (10 points) Two players have two boards $8 \times 8$ and $9 \times 9$, one by one they put rooks on these boards such that none of the rooks attack each other (on each turn a player can put a rook on only one board). Who is the winner in this game.

**Solution:** Note that this game is a sum of two games $G_1$ (this game on $9 \times 9$ board) and $G_2$ (this game on $8 \times 8$ board). During the first review session we proved that in $G_1$ the first player wins i.e. the initial position is an N-position, we also proved that in $G_2$ the second player wins i.e. the initial position is a P-position. Note that it means that the value of the Grundy function for $G_1$ is equal to $x \neq 0$ and for $G_2$ it is equal to 0. Hence, by the theorem about the Grundy function of the sum of two games, the value of the Grundy function for the sum of games is equal to $x$. 
2. Compute the Grundy function for states of the subtraction game with two piles of chips where players may subtract 1, 2 or 5 chips from one of the piles on their turn.

**Solution:** In the previous homework we proved that $g(n) = n \mod 3$ where $g(n)$ is the Grundy function of this game with one pile consisting of $n$ pebbles. By the theorem about the Grundy function of the sum of two games $f(n, m) = g(n) \oplus g(m) = (n \mod 3) \oplus (m \mod 3)$. 
Let $G_1$ be the subtraction game where on their turn a player may remove 1 or 2 coins, and where there are 10 coins initially. Let $G_2$ be the game of Nim with three piles, of sizes 1, 6, 7. List all winning moves in $G_1 + G_2$

**Solution:** Let $f$ be the Grundy function for $G_1$ and $g$ be the Grundy function for $G_2$. Let us compute $f(n)$ for $n \leq 10$:

- $f(0) = 0$,
- $f(1) = \text{mex}\{f(0)\} = 1$,
- $f(2) = \text{mex}\{f(0), f(1)\} = 2$,
- $f(3) = \text{mex}\{f(1), f(2)\} = 0$,
- $f(4) = \text{mex}\{f(2), f(3)\} = 1$,
- $f(5) = \text{mex}\{f(3), f(4)\} = 2$,
- $f(6) = \text{mex}\{f(4), f(5)\} = 0$,
- $f(7) = \text{mex}\{f(5), f(6)\} = 1$,
- $f(8) = \text{mex}\{f(6), f(7)\} = 2$,
- $f(9) = \text{mex}\{f(7), f(8)\} = 0$,
- $f(10) = \text{mex}\{f(8), f(9)\} = 1$.

It is also obvious that $g(1, 6, 7) = 1 \oplus 6 \oplus 7 = 0$. Hence, if $h$ is a Grundy function of $G_1 + G_2$, then $h(10, (1, 6, 7)) = 1$.

We need to find all the possible moves to P-positions (since if we go to an N-position we will lose). Since the value of $h$ in P-positions is equal to 0, there are only 3 moves from this position to a P-position:

1. we may remove one pebble in the first game,
2. we may remove one pebble from the first pile in the second game, and
3. we may remove one pebble from the last pile in the second game.