Note that every statement in the homework should be proved. The only exceptions are statements that were proven in previous homework or midterms and statements proven earlier in the class.

1. Alice and Bob has several apples and bananas and they want to split these fruits. Both of them like both fruits, but value them differently. For Alice, 1 apple is exactly equivalent to 2 bananas. For Bob, 2 apples are exactly equivalent to 1 banana.

   Show that the way to split all the fruits is Pareto optimal if and only if
   
   • either Alice has no bananas
   • or Bob has no apples.

   **Solution:** Suppose Alice has $A_1$ apples, and $B_1$ bananas; Bob has $A_2$ apples, and $B_2$ bananas.

   First of all, let us show that if we split all the fruits in a Pareto optimal way, then either Alice has no bananas or Bob has no apples. Let us assume for the sake of contradiction that $B_1$ or $A_2$ are not zero. If $B_1 > 0$, then Alice can trade her bananas with Bob for his apples, then both of their satisfaction levels are higher, contradiction. Similarly, if $A_2 > 0$. Therefore, $B_1$ and $A_2$ are 0.

   Let us prove now that if Alice has no bananas or Bob has no apples, then the current state is Pareto optimal. The only trade possible is: Alice trades her apples for Bob’s bananas. But since Bob values bananas more than apples, and Alice values apples more than bananas, the trade will make both parties worse off. Thus this is a Pareto optimum.
2. Let us consider a modified game of Nim: on each turn a player may remove some number of pebbles from one pile or split this pile into two piles. Compute the Grundy function for this game for all the initial states with one pile.

**Solution:** Let \( g \) denotes the Sprague–Grundy function. We claim that \( g(4k+1) = 4k+1, g(4k+2) = 4k+2, g(4k+3) = 4k+4, \) and \( g(4k+4) = 4k+3, \) for all \( k \geq 0. \)

We prove the claim by indication. The base case:

- \( g(1) = 1; \)
- \( g(2) = \text{mex}\{g(0), g(1), g(1)\} = 2 \)
- \( g(3) = \text{mex}\{g(0), g(1), g(2), g(1), g(2)\} = 4 \)
- \( g(4) = \text{mex}\{g(0), g(1), g(2), g(3), g(1), g(3), g(2), g(2)\} = 3 \)

The inductive hypothesis: the pattern holds for \( n \leq 4k, \) for some integer \( k \) We need to show that the pattern holds for \( n \leq 4(k+1). \)

- Let \( n = 4k+1. \) The followers that consist of a single pile have Sprague–Grundy values \( \{4k, 4k-1, 4k-2, 4k-3, \ldots, 0\}. \) The followers that consist of two piles are: \( \{(4k, 1), (4k - 1, 2), (4k - 2, 3), (4k - 3, 4), \ldots, (2k + 1, 2n)\} \) by the Sprague–Grundy theorem those positions have Sprague–Grundy values: \( \{4(k - 1) \oplus 1, 4k \oplus 2, 4(k - 2) \oplus 3 \ldots \}. \) Observe that all those values are even, thus \( g(4n + 1) = 4n + 1. \)

- Let \( n = 4k+2. \) The followers that consist of a single pile have Sprague–Grundy values \( \{4k + 1, 4k, 4k-1, 4k-2, \ldots, 0\}. \) The followers that consist of two piles are: \( \{(4k+1, 1), (4k, 2), (4k - 1, 3), (4k - 2, 4), \ldots, (2k + 1, 2k + 1)\} \) by the Sprague–Grundy theorem those positions have Sprague–Grundy values: \( \{(4k+1) \oplus 1, (4k - 1) \oplus 2, 4k \oplus 4 \ldots \}. \) Observe that all those values are alternately divisible by 4 and odd, thus \( g(4k + 2) = 4k + 2. \)

- Let \( n = 4k+3. \) The followers that consist of a single pile have Sprague–Grundy values: \( \{4k + 2, 4k + 1, 4n, 4n - 1, 4k - 2, 4k - 3, \ldots, 0\}. \) The followers that consist of two piles are: \( \{(4k+2, 1), (4k + 1, 2), (4k, 3), (4k - 1, 4), \ldots, (2k + 2, 2k + 1)\} \) by the Sprague–Grundy theorem those positions have Sprague–Grundy values: \( \{(4k+2) \oplus 1, (4k+1) \oplus 2, (4k - 1) \oplus 3 \ldots \}. \) Observe that all those values are odd and notice that \( g(4k + 2, 1) = 4k + 3. \) Thus \( g(4k + 3) = 4k + 4 \)

- Let \( n = 4k+4. \) The followers that consist of a single pile have Sprague–Grundy values: \( \{4k + 4, 4k + 2, 4n, 4n, 4k - 1, 4k - 2, \ldots, 0\}. \) The followers that consist of two piles are: \( \{(4k+3, 1), (4k+2, 2), (4k+1, 3), (4k, 4), \ldots, (2k+2, 2k+2)\} \) by the Sprague–Grundy theorem those positions have Sprague–Grundy values: \( \{(4k+4) \oplus 1, (4k+2) \oplus 2, (4k+1) \oplus 3 \ldots \}. \) Observe that all those values are alternately equal to \( 1 \) (mod \( 4 \)) and even. Thus \( g(4k + 4) = 4k + 3. \)

Thus the pattern extends to all integer \( n \geq 0. \)