1. (50 points) Check all the correct statements.

□ The Nim position (6, 4, 8) is an P-position.

□ In the subtraction game where players may subtract 3 and 5 chips on their turn 8 is an N-position.

■ The binary representation of 21 is 10101.

■ Bitwise XOR of 110111 and 111011 is 001100.

□ Nim-sum of 125 and 90 is 41.

2. (10 points) Consider the subtraction game where players may subtract 2, 4 or 5 chips on their turn, identify the N and P positions.

Solution: We are going to prove that position \( n \) is a P-position iff the remainder of \( n \) modulo 7 belongs to \{0, 1\}. We prove this statement using the strong induction by \( n \).

The base case is for \( n \in \{0, 1, \ldots, 6\} \). Note that since \( n = 0 \) and \( n = 1 \) are terminal positions they are P-positions. From 2 we can go to 0 i.e. it is an N-position. Since there is a move from 3 to 1 it is an N-position etc. Hence, types of states can be represented by the following table.

<table>
<thead>
<tr>
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<th>P</th>
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<tbody>
<tr>
<td>0</td>
<td>P</td>
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Let us prove now the induction step: by induction hypothesis we know that for every \( k < n \): \( k \) is a P-position iff the remainder of \( k \) modulo 7 belongs to \{0, 1\}.

- If \( n \equiv 0 \pmod{7} \), then we can only go to \( n - 2 \), \( n - 4 \), and \( n - 5 \) and their reminder modulo 7 are 5, 3, and 2, respectively; hence, \( n \) is a P-position.
- If \( n \equiv 1 \pmod{7} \), then we can only go to \( n - 2 \), \( n - 4 \), and \( n - 5 \) and their reminder modulo 7 are 6, 4, and 3, respectively; hence, \( n \) is a P-position.
- If \( n \equiv 2 \pmod{7} \), then we can go to \( n - 2 \) and the reminder modulo 7 is 0; hence, \( n \) is an N-position.
- If \( n \equiv 3 \pmod{7} \), then we can only go to \( n - 2 \), and the reminder modulo 7 is 1; hence, \( n \) is an N-position.
- If \( n \equiv 4 \pmod{7} \), then we can go to \( n - 4 \) and the reminder modulo 7 is 0; hence, \( n \) is an N-position.
- If \( n \equiv 5 \pmod{7} \), then we can go to \( n - 5 \) and the reminder modulo 7 is 0; hence, \( n \) is a N-position.
- If \( n \equiv 6 \pmod{7} \), then we can go to \( n - 5 \) and the reminder modulo 7 is 1; hence, \( n \) is an N-position.
3. (10 points) Two players play the following game.

- Initially, there are several coins on a line, with all their coordinates being positive integers.
- On each step a player can move one coin to the left as far as they want but they cannot jump over another coin (they cannot move a coin to a non positive point).
- The game ends when one of them cannot make a move and the player that cannot make a move loses.

Describe all the initial states in which the first player wins.

**Solution:** It is possible to note that this game is equivalent to the staircase Nim.

However, let us analyze it from the scratch. Let the coordinates of the coins be $x_1, \ldots, x_n$. Note that the most important parameters of the game are distances between these coins, i.e. $y_1 = x_1$, $y_2 = x_2 - x_1 - 1$, $\ldots$, $y_n = x_n - x_{n-1} - 1$.

It is easy to see that the position is terminal iff $y_1 = 0, \ldots, y_n = 0$. Additionally, the position such that $y_1 = 0, \ldots, y_{n-1} = 0, y_n = d$ is an N-position for every $d \in \mathbb{N}_0$ since we may move the last coin to the left.

Let us prove that the position is a P-position iff $y_n \oplus y_{n-2} \oplus y_{n-4} \cdots = 0$.

To prove this we use induction on the sum of $y_i$. If all of them are zero it is a terminal position and hence it is a P-position.

Let us prove the induction step.

- If $y_n \oplus y_{n-2} \oplus y_{n-4} \cdots = 0$ we need to show that every move leads us to an N-position. Note that if we move $i$th coin we change only $y_i$ and $y_{i+1}$ and only one of them participate in the sum $y_n \oplus y_{n-2} \oplus y_{n-4} \cdots = 0$. Hence, all reachable positions are N-positions i.e. this position is a P-position.

- If $y_n \oplus y_{n-2} \oplus y_{n-4} \cdots \neq 0$ we need to show that we can go to a P-position. In class we proved that there is $d > 0$ and $k > 0$ such that $y_n \oplus y_{n-2} \oplus y_{n-4} \cdots y_{n-2k} - d \cdots = 0$. Hence, if we move the $(n-2k)$th coin we get a P-position. As a result, the current position is an N-position.
4. (10 points) Two players one by one put bishops on the chessboard such that every new bishop attacks at least one previously not attacked square. The player that can not make a move loses. Determine the winning strategy.

**Solution:** There is a strategy for the second player (Bob). On each step this player put a new bishop on the position symmetric to the position of the previous bishop (symmetric with respect to the middle line of the chessboard).

For example if the first player make the following move

we put our bishop like in this picture.

Note that if Alice’s bishop attacks a square $P$ that was not attacked before, then Bob’s bishop attacks the symmetric to $P$ position $Q$ since after every our move the picture is symmetric and also note that $P$ and $Q$ have different colors; hence, Alice’s bishop does not attack $Q$. Hence, if Alice have a move, then so do Bob.