1. (50 points) Check all the correct statements.

■ The Nim position (6, 5, 7) is an N-position.

□ In the subtraction game where players may subtract 2 and 3 chips on their turn 5 is an N-position.

□ The binary representation of 38 is 100100.

■ Bitwise XOR of 100111 and 111111 is 011000.

■ Nim-sum of 14 and 21 is 27.
2. (10 points) Two players one by one put bishops on the chessboard such that none of them attack each other. Determine the winning strategy.

**Use symmetric strategies.**

**Solution:** There is a strategy for the second player (Bob). On each step this player put a new bishop on the position symmetric to the position of the previous bishop (symmetric with respect to the middle line of the chessboard).

For example if the first player make the following move

![Chessboard Diagram](image1)

we put our bishop like in this picture.

![Chessboard Diagram](image2)

Note that if Alice’s bishop was not attacked, then Bob’s bishop is not attacked neither since after every our move the picture is symmetric and Alice’s bishop is on the same line as Bob’s. Hence, if Alice have a move, then so do Bob.
3. (10 points) Consider the Misère subtraction game where players may subtract 1, 2 or 5 chips on their turn, identify the N and P positions.

**Solution:** We are going to prove that a position $n$ is a P-position iff the reminder of $n$ modulo 3 is equal to 1. We prove this statement using the strong induction by $n$.

The base case is for $n \in \{0, 1, \ldots, 6\}$. Note that since $n = 0$ is a terminal position it is an N-position. From 1 we can go only to 0 i.e. it is a P-position. Since there is a move from $i$ to 1 it is an N-state etc.

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Let us prove now the induction step: by induction hypothesis we know that for every $k < n$: $k$ is a P-position iff the reminder of $k$ modulo 3 is equal to 1.

- If $n \equiv 0 \pmod{3}$, then we can go to $n - 2$ and the reminder modulo 3 is 1; hence, $n$ is an N-position.
- If $n \equiv 1 \pmod{3}$, then we can only go to $n - 1$, $n - 2$, and $n - 5$ and the reminders modulo 3 are 0, 2, and 2; hence, $n$ is a P-position.
- If $n \equiv 2 \pmod{3}$, then we can go to $n - 1$ and the reminder modulo 3 is 1; hence, $n$ is an N-position.
4. (10 points) Two players play the following game: on each step they move a rook up or to the right (on any number of squares); the rook begins on a1.
Determine who wins in this combinatorial game.

**Solution:** There are several solutions for this game, the first and the simplest solution is to note that this game is the same as the game of Nim with two piles made of 8 pebbles each.

Another solution is to return the rook to the main diagonal after each step of the first player. Since the game ends only when we reach the rightmost top corner, the first player can not win.

For example if the first player do this move

![Chessboard with rook on a1](image1)

we need to return the rook back to the main diagonal, like this.

![Chessboard with rook back to main diagonal](image2)